MATH 3260H Coordinatization III: Ternary Rings

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Recall: We defined a ternary ring after introducing coordinates in a projective plane via

$$y = T(m,x,b) \iff (x,y)\, I\, [m,b].$$

("$mx+b$" (combines $\cdot$ & $+$))

and then we defined $+$ & $\cdot$ via

$$c+d = T(1,c,d)$$

$$\& \quad c \cdot d = T(c,d,0).$$

What algebraic can we guarantee that a ternary ring has?

Def'n: A (planar) ternary ring is a set of elements R (including 0&1) together with a three-place operation $T: R^3 \to R$ satisfying:

(1) For all $a, b \in R$, $T(a,0,b) = T(0,a,b) = b$.

(2) For all $a \in R$, $T(a,1,0) = T(1,a,0) = a$.

(3) Given any (x,y) & (u,v) from R^2 with $x \neq u$, there is an unique $(m,b) \in R^2$ such that
$$y = T(m,x,b) \quad \text{and} \quad v = T(m,u,b).$$

(4) For all $x, y, m \in R$, there is an unique $b \in R$ such that $y = T(m,x,b)$.

(5) For all $m, b, n, c \in R$ with $m \neq n$, there are unique $x, y \in R$ such that $T(m,x,b) = y = T(n,x,c)$.

<u>Theorem:</u> If we define a ternary ring from the coordinates of a projective plane, then it is indeed a planar ternary ring, <u>ie</u> it satisfies conditions (1)-(5) of the def'n.

Conversely, given a planar ternary ring you (re)construct a projective plane via the affine part of the coordinate system using (again) the relation

$$(x,y)\ I\ [m,b] \iff y = T(m,x,b).$$

<u>proof:</u> On an assignment... //

If ~~we~~ define + & · from T as before, what algebraic properties do they satisfy?

Short version: much less than we'd like, as we cannot even guarantee that $T(m,x,b) = (m\cdot x) + b$.

From our definition of a planar ternary ring we do get that $a \cdot 0 = 0 \cdot a = 0$ and $a \cdot 1 = 1 \cdot a = a$.

<u>Def'n:</u> A non-empty set L with a binary operation $\circ : L^2 \to L$ is a <u>loop</u> if it satisfies the following conditions:

1) For all $a, b \in L$, there is an unique $x \in L$ such that $a \circ x = b$.
2) For all $a, b \in L$, there is an unique $y \in L$ such that $y \circ a = b$.

Given a & b, we could have $x \neq y$ ...

3) There is an element $e \in L$, such that for all $a \in L$, $a \circ e = e \circ a = a$.

<u>Theorem:</u> If ~~[illegible]~~ (R, T) is a planar ternary ring, then $(R, +)$ & ~~$(R, \circ)$~~ $(R \setminus \{0\}, \circ)$ are loops.

<u>proof:</u> For multiplization. (for addition it's pretty similar).

(1) We need to check that for all $a, b \in R \setminus \{0\}$ there is an $x \in R\setminus\{0\}$ s.t. $a \cdot x = b$.

<u>ie</u> we need to show that there is an unique x s.t. $T(a,x,0) = b$

Consider the cases that $a = 0$ & $a \neq 0$.

If $a = 0$, by condition (1) of ternary rings

$$T(0,0,0) = 0$$

We'll do this next time when my brain is back ...