A Problem Course on Projective Planes

Version 0.4

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1991 *Mathematics Subject Classification.* 51
Key words and phrases. projective plane

To Prof. F.A. Sherk, who got me interested in this topic.

Abstract. This is a text for a problem-oriented course on projective planes.

This work was typeset with LaTeX, using the AMS-LaTeX and AMSFonts packages of the American Mathematical Society.

Contents

Preface

This book is intended to be the basis for a problem-oriented course on projective planes for students with a modicum of mathematical sophistication. It covers the basic definitions of affine and projective planes, some methods of constructing them, the introduction of coordinates, collineations, and the basics of the relationships between the geometry of the plane, the algebraic properties of possible coordinate systems, and the properties of its collineation group.

In keeping with the modified Moore-method, this book supplies definitions, problems, and statements of results, along with some explanations, examples, hints, and sample solutions. The intent is for the students, individually or in groups, to learn the material by solving the problems and proving the results for themselves. Besides constructive criticism, it will probably be necessary for the instructor to supply further hints or direct the students to other sources from time to time. Just how this text is used will, of course, depend on the instructor and students in question. However, it is probably *not* appropriate for a conventional lecture-based course or for a large class.

The material presented in this volume is somewhat stripped-down. Various concepts and topics are given very short shrift or omitted entirely.[1] Instructors might consider having students do projects on additional material if they wish to to cover it.

Acknowledgements. Various people and institutions deserve credit for this work:

- All the people who developed the subject.
- My teachers and colleagues, especially Prof. F.A. Sherk of the University of Toronto, whose geometry courses, and an NSERC summer research project he supervised, got me interested in this subject.
- The students at Trent University who suffered, suffer, and will suffer through assorted versions of this text.

[1]Future versions of both volumes may include more – or less! – material. Feel free to send suggestions, corrections, criticisms, and the like — I'll feel free to ignore them or use them.

- Trent University and the taxpayers of the Province of Ontario, who paid my salary.
- All the people and organizations who developed the software and hardware with which this book was prepared.
- Anyone else I've missed.

Any blame properly accrues to the author.

Availability. The URL of the home page for *A Problem Course on Projective Planes*, with links to LaTeXand Portable Document Format (pdf) files of the latest available release, is:

- `http://euclid.trentu.ca/math/sb/pcpp/`

Please note that in addition to LaTeX you will need the AMS-LaTeX and AMSFonts packages to typeset and print the LaTeX source.

If you have any problems, feel free to contact the author at the addresses given on the title page, preferably by e-mail, for assistance.

CHAPTER 1

Basics

Incidence structures and configurations. The geometrical notion that we will focus on, to the exclusion of notions like distance and angle, is that of *incidence*, *i.e.* the relation of points being on lines or lines passing through points.

DEFINITION 1.1. An *incidence structure* is a triple $(\mathcal{P}, \mathcal{L}, I)$, consisting of a set $\mathcal{P}$ of *points*, a set $\mathcal{L}$ of *lines*, and a relation I of *incidence* between elements of $\mathcal{P}$ and elements of $\mathcal{L}$.

If a point $P \in \mathcal{P}$ is incident with a line $\ell \in \mathcal{L}$, usually written as $PI\ell$, P is often said to be *on* ℓ, and ℓ is often said to *pass through* P. Two lines which are both incident with a particular point are usually said to *intersect*, or to be *coincident*, at that point. Two points which are both incident with the same line are sometimes said to be *joined* or *connected* by that line.

EXAMPLE 1.1. The geometry of points and great circles on a sphere gives an incidence structure in which any two different lines intersect in exactly two points. The points are the points on (the surface of) a sphere and the lines are the great circles of the sphere; incidence works in the obvious way. (See Figure 1.)

EXAMPLE 1.2. Figure 2 illustrates a finite incidence structure, the *Fano configuration*, with seven points and seven lines. The Fano configuration is the smallest example of a projective plane and will occur frequently as an example.

Note that the relation of incidence in a general incidence structure may be completely arbitrary. Points may or may not be connected by lines, lines may or may not intersect, there may be multiple points of intersection for a given pair of lines, and there may be multiple lines joining a given pair points. We will stick to incidence structures which avoid some of these pathologies:

DEFINITION 1.2. An *configuration* is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ satisfying the following conditions:

- Any two distinct points are incident with at most one line.

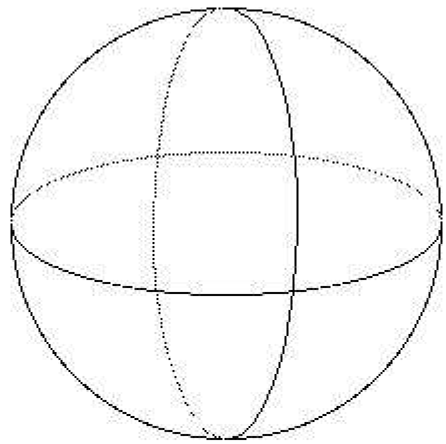

FIGURE 1. Points and great circles on a sphere

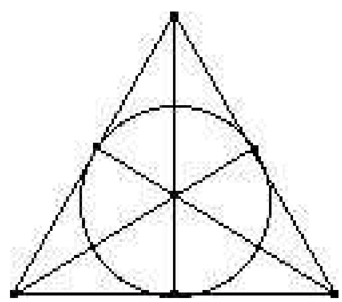

FIGURE 2. Fano configuration

- Any two distinct lines are incident with at most one point.

Interesting incidence structures that are not configurations do turn up, such as the geometry of great circles on a sphere; we just won't be studying them...

PROBLEM 1.1. *Show that the geometry of points and great circles on sphere (Example 1.1) is not a configuration.*

PROBLEM 1.2. *Show that the Fano configuration (Example 1.2) is a configuration.*

Affine planes. The incidence structure most likely to turn up in basic geometry is the Euclidean plane, which is an example of an affine plane. Affine planes are of interest to us because of their close connections to projective planes.

DEFINITION 1.3. An *affine plane* is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$, such that $\mathcal{P}$ and $\mathcal{L}$ are non-empty sets, and satisfying the following axioms:

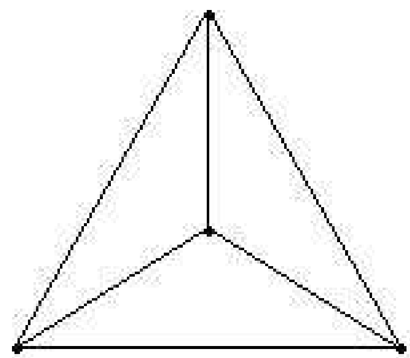

FIGURE 3. A finite affine plane

- **AI:** Any two distinct points are incident with an unique line.
- **AII:** Given a point P and a line ℓ not incident with P, there is an unique line m incident with P which has no point in common with ℓ.
- **AIII:** There exist three points which are not incident with the same line.

Two lines in an affine plane are said to be *parallel* if they have no point in common, *i.e.* if they do not intersect.

EXAMPLE 1.3. Figure 3 illustrates a finite affine plane with four points and six lines. This is the smallest example of an affine plane and, as we shall see, it is closely related to the Fano configuration.

PROBLEM 1.3. *Draw an affine plane which has three points on every line. How many points and lines does it have in total?*

EXAMPLE 1.4. The *real affine plane* or *Euclidean plane* can be thought of as $\mathbb{R}^2$ with the usual Cartesian coordinate system and the usual points, lines, and incidence relation. It is sometimes denoted by $AG(2, \mathbb{R})$.

PROBLEM 1.4. *Verify that the Euclidean plane is indeed an affine plane.*

Some key facts about affine planes are summarized in the following result.

PROPOSITION 1.5. *Suppose* $(\mathcal{P}, \mathcal{L}, I)$ *is an affine plane. Then the lines in* $\mathcal{L}$ *can be partitioned into* parallel classes *such that all the lines in each class are mutually parallel, any two lines from different parallel classes intersect, and any point* $P \in \mathcal{P}$ *is on exactly one line of each parallel class.*

Projective planes. Of course, our object is to study projective planes.

DEFINITION 1.4. A *projective plane* is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$, such that $\mathcal{P}$ and $\mathcal{L}$ are non-empty sets, and satisfying the following axioms:

I: Any two distinct points are incident with an unique line.
II: Any two distinct lines are incident with an unique point.
III: There exist four points such that no three of them are incident with the same line.

If a point P is incident with a line ℓ, usually written as $PI\ell$, P is often said to be *on* ℓ, and ℓ is often said to *pass through* P.

EXAMPLE 1.5. The Fano configuration defined in Example 1.2 is a projective plane.

PROBLEM 1.6. *Show that the Fano configuration is indeed a projective plane.*

PROBLEM 1.7. *Draw a projective plane which has four points on every line. How many points and lines does it have in total?*

EXAMPLE 1.6. Suppose $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^3$ is the three-dimensional vector space over $\mathbb{R}$. The *real projective plane*, sometimes denoted by $PG(2, \mathbb{R})$, is the incidence structure $(\mathcal{P}, \mathcal{L}, I)$ defined as follows:

- $\mathcal{P}$ is the set of one-dimensional subspaces of $\mathbb{R}^3$ (*i.e.* the lines through the origin).
- $\mathcal{L}$ is the set of two-dimensional subspaces of $\mathbb{R}^3$ (*i.e.* the planes through the origin).
- For all points $P \in \mathcal{P}$ and lines $\ell \in \mathcal{L}$, $PI\ell$ if and only if $P \subset L$.

PROBLEM 1.8. *Show that the real projective plane is indeed a projective plane.*

Degenerate planes. An incidence structure that satisfies axioms I and II, but not III, for a projective plane is sometimes called a *degenerate plane*. Figure 4 gives a couple of examples.

PROBLEM 1.9. *Find all the degenerate planes.*

Some basic properties of projective planes.

PROPOSITION 1.10. *Suppose $(\mathcal{P}, \mathcal{L}, I)$ is a projective plane. Then:*

(1) *Every line in $\mathcal{L}$ is incident with at least three points in $\mathcal{P}$.*
(2) *Every point in $\mathcal{P}$ is incident with at least three lines in $\mathcal{L}$.*

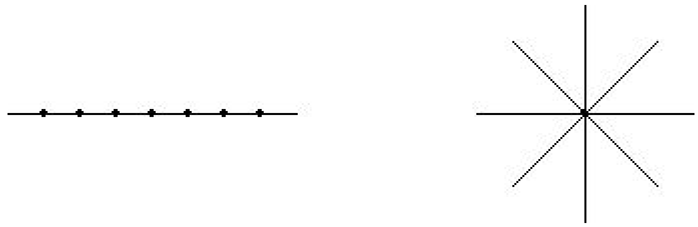

FIGURE 4. A pair of degenerate planes

(3) *Every projective plane has at least seven points and seven lines.*

Note that this result means that the Fano configuration is as small as a projective plane can get, *i.e.* it has as few points and lines as possible.

PROPOSITION 1.11. *Every line of a projective plane is incident with just as many points as any other line.*

DEFINITION 1.5. A projective plane $(\mathcal{P}, \mathcal{L}, I)$ is said to be *finite* if $\mathcal{P}$ is finite. In this case, the plane is said to be of order n if every line contains exactly $n+1$ points. A projective plane is said to be *infinite* if it is not finite.

PROPOSITION 1.12. *Suppose* $(\mathcal{P}, \mathcal{L}, I)$ *is a finite projective plane of order* n. *Then:*

(1) $n+1$ *lines pass through every point in* $\mathcal{P}$.
(2) $\mathcal{P}$ *has exactly* n^2+n+1 *points.*
(3) $\mathcal{L}$ *has exactly* n^2+n+1 *lines.*

Thanks to axiom II, there are no parallel lines in a projective plane. However, the axioms for projective planes have one interesting property that their counterparts for affine planes do not, namely *duality*. In particular, we have the following:

PROPOSITION 1.13. *Suppose* $\pi = (\mathcal{P}, \mathcal{L}, I)$ *is a projective plane. If we define* $I' \subset \mathcal{L} \times \mathcal{P}$ *by saying that* $\ell I' P$ *exactly when* $P I \ell$, *then* $\pi' = (\mathcal{L}, \mathcal{P}, I')$ *is also a projective plane.*

More generally, anything that can be proven from axioms I–III about points and lines in a projective plane can also be proven if we interchange the roles of lines and points.

PROBLEM 1.14. *Make the assertion in the previous paragraph precise and prove it.*

Homomorphisms and isomorphisms. From time to time we will need to know when two incidence structures, especially two planes, are essentially the same.

DEFINITION 1.6. Suppose $\pi_1 = (\mathcal{P}_1, \mathcal{L}_1, I_1)$ and $\pi_2 = (\mathcal{P}_2, \mathcal{L}_2, I_2)$ are incidence structures. A *homomorphism* from π_1 to π_2 is a function $\varphi : (\mathcal{P}_1 \cup \mathcal{L}_1) \to (\mathcal{P}_2 \cup \mathcal{L}_2)$ such that

- for all $P \in \mathcal{P}_1$, $\varphi(P) \in \mathcal{P}_2$,
- for all $\ell \in \mathcal{L}_1$, $\varphi(\ell) \in \mathcal{L}_2$, and
- for all $P \in \mathcal{P}_1$ and all $\ell \in \mathcal{L}_1$, if $P I_1 \ell$, then $\varphi(P) I_2 \varphi(\ell)$.

A homomorphism $\varphi : \pi_1 \to \pi_2$ is an *isomorphism* if φ is also 1–1 and onto. π_1 and π_2 are then said to be *isomorphic*, often written as $\pi_1 \cong \pi_2$.

PROBLEM 1.15. *$PG(2, \mathbb{Q})$ is defined from the rational numbers in the same way that $PG(2, \mathbb{R})$ is defined from the real numbers. Show that there is an 1–1 homomorphism of projective planes from $PG(2, \mathbb{Q})$ to $PG(2, \mathbb{R})$, but that these projective planes are not isomorphic.*

PROBLEM 1.16. *Suppose φ is an isomorphism of projective planes from $\pi_1 = (\mathcal{P}_1, \mathcal{L}_1, I_1)$ to $\pi_2 = (\mathcal{P}_2, \mathcal{L}_2, I_2)$. Show that φ^{-1} is an isomorphism from π_2 to π_1.*

CHAPTER 2

Some constructions

Affine planes via coordinates. The idea used in Example 1.4, *affine coordinates*, can be readily generalized to any field, not just the real numbers.

Problem 2.1. *If F is a field, $AG(2, F)$ is the the incidence structure built from F in the same way that $AG(2, \mathbb{R})$ is from $\mathbb{R}$. Verify that $AG(2, F)$ is an affine plane.*

Problem 2.2. *Show that if $F = GF(2)$ is the two-element field, then $AG(2, F)$ is isomorphic to the finite affine plane described in Example 1.3.*

In fact, affine coordinates still work even if F isn't quite a field.

Problem 2.3. *Suppose S is a* skew field*; that is, S satisfies all the field axioms except possibly for commutativity of multiplication. Show that $AG(2, S)$ is an affine plane.*

There are, however, affine planes which are *not* coordinatized by fields or skew fields. One of the classic examples of such an affine plane is the following:

Example 2.1. The *Moulton plane* is the affine plane $(\mathcal{P}, \mathcal{L}, I)$ defined as follows:

(1) The points of $\mathcal{P}$ are the points of $\mathbb{R}^2$.
(2) The lines of $\mathcal{L}$ are sets of all points (x, y) satisfying one of the following conditions:
   (a) $x = c$ for some fixed $c \in \mathbb{R}$.
   (b) $y = c$ for some fixed $c \in \mathbb{R}$.
   (c) $y = m(x - a) \cdot f(y, m) + b$ for some fixed $m, a, b \in \mathbb{R}$, where

$$f(y, m) = \begin{cases} 1 & \text{if } m \leq 0 \text{ or } y \leq 0 \\ \frac{1}{2} & \text{if } m > 0 \text{ and } y > 0 \end{cases}$$

(3) Incidence is inclusion, *i.e.* $PI\ell$ if $P \in \ell$.

See Figure 1 for a picture of some lines in the Moulton plane.

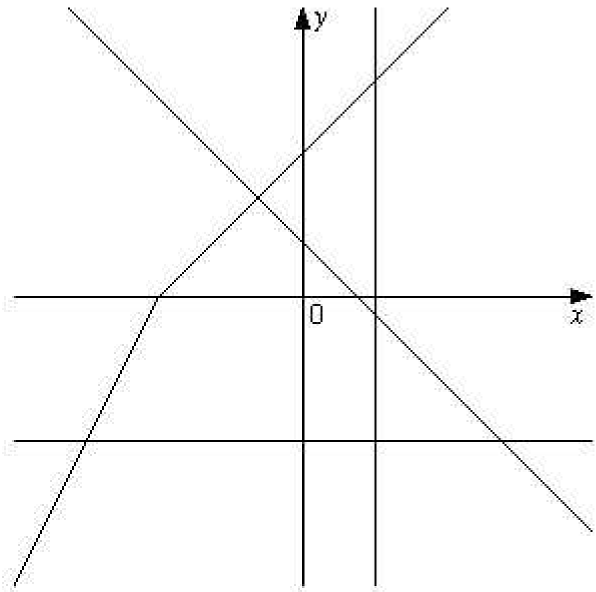

FIGURE 1. Lines in the Moulton plane

PROBLEM 2.4. *Verify that the Moulton plane is indeed an affine plane.*

This construction of the Moulton plane is an example of one approach to constructing examples and counterexamples: start with a plane – in this case $AG(2, \mathbb{R})$ – and redefine which sets of points are the lines.

Projective planes via coordinates. As with its counterpart for affine planes, Example 1.4, Example 1.6 can be readily generalized.

PROBLEM 2.5. *If F is an arbitrary field, $PG(2, F)$ is defined from F in the same way that $PG(2, \mathbb{R})$ is defined from $\mathbb{R}$. Show that $PG(2, F)$ is a projective plane.*

PROBLEM 2.6. *Show that if $F = GF(2)$ is the two-element field, then $PG(2, F)$ is essentially the Fano configuration.*

As with the construction of $AG(2, F)$, the definition of $PG(2, F)$ still makes sense and gives a projective plane if F is skew field instead of a field. One can also construct projective planes which are not coordinatized by fields or skew fields, as we shall see a little later.

A standard method of building a projective plane from a field (or something close to one) is to use *projective coordinates*. This is really the same technique used in Example 1.6 and above, just written out in a different way.

DEFINITION 2.1. Suppose F is a field. Define an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ as follows:

(1) $\mathcal{P}$ consists of triples $(x, y, z) \in F^3 \setminus \{(0,0,0)\}$, subject to the condition that two triples (x, y, z) and (u, v, w) represent the same point if there is a non-zero $\lambda \in F$ such that $x = \lambda u$, $y = \lambda v$, and $z = \lambda w$.
(2) $\mathcal{L}$ consists of triples $[a, b, c] \in F^3 \setminus \{(0,0,0)\}$, subject to the condition that two triples $[a, b, c]$ and $[d, e, f]$ represent the same line if there is a non-zero $\lambda \in F$ such that $a = \lambda d$, $b = \lambda e$, and $c = \lambda f$.
(3) $(x, y, z)I[a, b, c] \iff ax + by + cz = 0$.

PROBLEM 2.7. *Verify that the procedure given above does define a projective plane and show that it is isomorphic to $PG(2, F)$ as previously defined.*

Another way of using a field (or something close to one) to build a projective plane by way of a coordinate system is an extension of the method for building an affine plane via affine coordinates.

DEFINITION 2.2. Suppose F is a field and ∞ is a symbol which is not in F. Define an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ as follows:

(1) The points in $\mathcal{P}$ include:
    - all pairs $(x, y) \in F^2$,
    - all singletons (m) for $m \in F$, and
    - the singleton (∞).

    That is, as a set, $\mathcal{P} = F^2 \cup \{\, (m) \mid a \in F \,\} \cup \{(\infty)\}$.
(2) The lines in $\mathcal{L}$ consist of the following sets of points:
    - for each m and b in F, the line given by $y = mx + b$, $[m, b] = \{\, (x, y) \in F^2 \mid y = mx + b \,\} \cup \{(m)\}$,
    - for each $c \in F$, the line given by $x = c$, $[c] = \{\, (x, c) \mid x \in F \,\} \cup \{(\infty)\}$, and
    - the *line at infinity*, $[\infty] = \{\, (m) \mid m \in F \,\} \cup \{(\infty)\}$.
(3) $PI\ell$ if and only if $P \in \ell$.

EXAMPLE 2.2. We can turn the Euclidean plane, $AG(2, \mathbb{R})$, into a projective plane by adding

- a point (m) for each $m \in \mathbb{R}$ where all the lines with slope m meet,
- a point (∞) where all the vertical lines meet, and

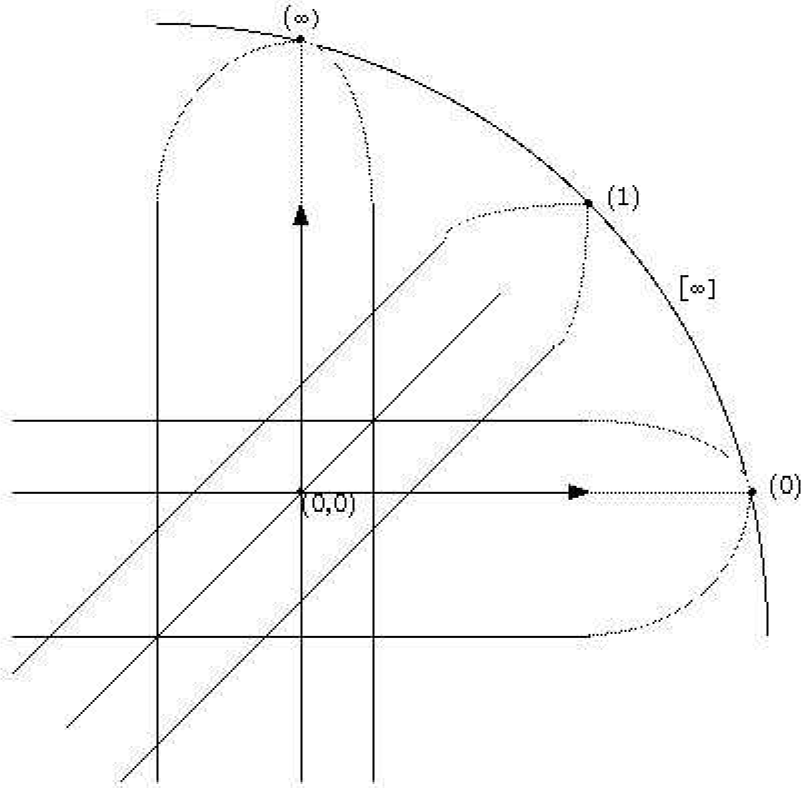

FIGURE 2. Extended affine coordinates

- a line at infinity, $[\infty]$, passing through all the new points (and no others),

as in Figure 2.

PROBLEM 2.8. *Verify that the construction does define a projective plane and that it is isomorphic to $PG(2, F)$.*

This method of *extended affine coordinates* is an application of a more general procedure of constructing affine and projective planes from each other, described below. It is worth independent mention in part because the method we will eventually use to construct a coordinate system for an arbitrary projective plane creates an extended affine coordinate system for that plane, though the coordinates do not necessarily form anything close to a field.

Affine from projective planes and *vice versa*. It turns out that one can easily construct an affine plane from a given projective plane, or *vice versa*, simply by deleting, or adding, a line and all the points on it.

THEOREM 2.9. *Suppose* $(\mathcal{P}, \mathcal{L}, I)$ *is a projective plane and let* ℓ *be any fixed line of* $\mathcal{L}$. *Define an incidence structure* $(\mathcal{P}', \mathcal{L}', I')$ *as follows:*

(1) $\mathcal{P}' = \mathcal{P} \setminus \ell$, i.e. *the points of* $\mathcal{P}'$ *are the points of* $\mathcal{P}$ *except for those on* ℓ.
(2) $\mathcal{L}' = \mathcal{L} \setminus \{\ell\}$, i.e. *the lines of* $\mathcal{L}'$ *are the lines of* $\mathcal{L}$ *except for* ℓ.
(3) $I' = I \cap (\mathcal{P}' \times \mathcal{L}')$, i.e. *incidence doesn't change, apart from there being fewer points and lines.*

Then $(\mathcal{P}', \mathcal{L}', I')$ *is an affine plane.*

PROBLEM 2.10. *Show that deleting a line and all the points on it from the Fano configuration gives the affine plane of Example 1.3.*

THEOREM 2.11. *Suppose* $(\mathcal{P}, \mathcal{L}, I)$ *is an affine plane. Define an incidence structure* $(\mathcal{P}^*, \mathcal{L}^*, I^*)$ *as follows:*

(1) *The points of* $\mathcal{P}^*$ *are all the points of* $\mathcal{P}$, *together with one* ideal point P_C *for each parallel class* C *of lines in* $\mathcal{L}$.
(2) *The lines of* $\mathcal{L}^*$ *are all the lines of* $\mathcal{L}$, *plus one* ideal line *or* line at infinity, ℓ_∞.
(3) I^* *is defined as follows:*
    (a) $PI^*\ell$ *if* $PI\ell$.
    (b) $P_C I^*\ell$ *for every line* ℓ *in the parallel class* C.
    (c) *For each parallel class* C, $P_C I^* \ell_\infty$.

Then $(\mathcal{P}^*, \mathcal{L}^*, I^*)$ *is a projective plane.*

PROBLEM 2.12. *Show that adding a line at infinity to the affine plane of Example 1.3 gives the Fano configuration.*

PROBLEM 2.13. *Show that if* F *is field, then* $AG(2, F)$ *and* $PG(2, F)$ *are corresponding affine and projective planes in the sense of the procedures given in the above theorems.*

Free completion. The methods described so far all rely on having a suitable algebraic structure or an affine plane in order to construct a projective plane. There is a fairly elementary technique for constructing a projective plane starting from nothing but a suitable configuration.

DEFINITION 2.3. Suppose $\mathcal{C}_0 = (\mathcal{P}_0, \mathcal{L}_0, I_0)$ is a configuration. For each $n \geq 0$, given a configuration $\mathcal{C}_n = (\mathcal{P}_n, \mathcal{L}_n, I_n)$, define the configuration $\mathcal{C}_{n+1} = (\mathcal{P}_{n+1}, \mathcal{L}_{n+1}, I_{n+1})$ as follows:

(1) $\mathcal{P}_{n+1}$ includes all the points of $\mathcal{P}_n$, together with a new and distinct point of intersection for every pair of lines in $\mathcal{L}_n$ which do not already have a common point of intersection in $\mathcal{C}_n$.

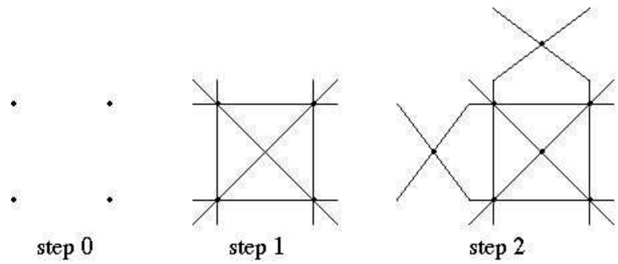

FIGURE 3. Free completion of a quadrangle

(2) $\mathcal{L}_{n+1}$ includes all the lines of $\mathcal{L}_n$, together with a new and distinct line joining each pair of points in $\mathcal{P}_n$ which do not already have a line joining them in $\mathcal{C}_n$.
(3) I_{n+1} is I_n, plus the incidences involving the added points and lines described above.

Let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$, $\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n$, and $I = \bigcup_{n=0}^{\infty} I_n$. Then the incidence structure $\pi = (\mathcal{P}, \mathcal{L}, I)$ is the *free completion* of the configuration $\mathcal{C}_0 = (\mathcal{P}_0, \mathcal{L}_0, I_0)$.

To get a projective plane using this process, we need to assume a little about the configuration to ensure that axiom III holds.

DEFINITION 2.4. A *quadrangle* is a set of four points, no three of which are incident with the same line, and a *quadrilateral* is a set of four lines, no three of which are incident with a single point.

EXAMPLE 2.3. The first two configurations obtained in the process of freely completing a quadrangle are shown in Figure 3.

THEOREM 2.14. *The free completion of a configuration containing either a quadrangle or a quadrilateral is a projective plane.*

PROBLEM 2.15. *Show that the free completion of a (non-empty) configuration which does not contain either a quadrangle or a quadrilateral need not be a projective plane.*

Subplanes and Completion. If a given configuration is contained in a projective plane, one can also complete it using the incidence relation of the plane, in which case its completion is a projective plane contained within the original one. To consider this properly we first need to introduce the notion of a subplane of a projective plane.

DEFINITION 2.5. Suppose $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane. A *subplane* of π is a structure $\pi_0 = (\mathcal{P}_0, \mathcal{L}_0, I_0)$ where

(1) $\mathcal{P}_0 \subseteq \mathcal{P}$,

(2) $\mathcal{L}_0 \subseteq \mathcal{L}$,
(3) $I_0 = I \cap (\mathcal{P}_0 \times \mathcal{L}_0)$, and
(4) π_0 is itself a projective plane.

PROBLEM 2.16. *Show that $PG(2,\mathbb{Q})$ is a subplane of $PG(2,\mathbb{R})$.*

DEFINITION 2.6. Suppose $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane and $\mathcal{C} \subseteq \mathcal{P} \cup \mathcal{L}$ is a configuration in π. (Points and lines of $\mathcal{C}$ are incident in $\mathcal{C}$ exactly as they are incident with each other in π.) Define a sequence of configurations, $\mathcal{C}_n$ for $n \in \mathbb{N}$, as follows:

- $\mathcal{C}_0 = \mathcal{C}$
- $\mathcal{C}_{n+1} = \mathcal{C}_n$, together will all lines in $\mathcal{L}$ joining pairs of points in $\mathcal{C}_n$ and all points in $\mathcal{P}$ where two line of $\mathcal{C}_n$ intersect.

Then the structure $\mathcal{C}' = (\mathcal{P}', \mathcal{L}', I')$, where

(1) $\mathcal{P}' = \mathcal{P} \cap \bigcup_{n=0}^{\infty} \mathcal{C}_n$,
(2) $\mathcal{L}' = \mathcal{L} \cap \bigcup_{n=0}^{\infty} \mathcal{C}_n$, and
(3) $I' = I \cap (\mathcal{P}' \times \mathcal{L}')$,

is the *completion* of $\mathcal{C}$ in π.

The procedure described above can, of course, be carried out within any incidence structure, but we will stick to doing so within projective planes.

PROBLEM 2.17. *Show that if $\mathcal{C}$ is a quadrangle within the Fano configuration, then its completion is the whole Fano configuration.*

PROPOSITION 2.18. *Suppose $\mathcal{C}$ is a configuration in a projective plane π which contains a quadrangle or a quadrilateral. Then the completion of $\mathcal{C}$ in π is a subplane of π. (It is usually called the* subplane generated by *$\mathcal{C}$.)*

PROBLEM 2.19. *Find a quadrangle in $PG(2,\mathbb{R})$ whose completion is $PG(2,\mathbb{Q})$.*

Homomorphisms supply another reason for considering subplanes.

PROPOSITION 2.20. *Suppose φ is a 1–1 homomorphism of projective planes from π_1 to π_2. Then the image of φ is a subplane of π_2.*

PROBLEM 2.21. *Suppose φ is a homomorphism of projective planes from π_1 to π_2. If φ is not 1–1, does the image of φ still have to be a subplane of π_2?*

CHAPTER 3

Collineations

Collineations. A lot of the real action in the study of projective planes involves the isomorphisms from a plane to itself, that is, the automorphisms or collineations of a plane. As we shall see in later chapters, the properties of the automorphisms of a plane are intimately tied to the geometrical properties of the plane and to the properties of any algebraic stucture which can be used to coordinatize the plane.

DEFINITION 3.1. A *collineation* of a projective plane is an isomorphism of the plane to itself. (That is, a collineation is an automorphism of a projective plane.)

It is common to write the image of a point, $\gamma(P)$, under the collineation γ as P^γ, and similarly for lines. (*i.e.* $\gamma(P) = P^\gamma$ and $\gamma(\ell) = \ell^\gamma$.) A collineation γ is said to *fix* a point P (respectively, a line ℓ) if $P^\gamma = P$ (respectively, $\ell^\gamma = \ell$. γ is the *identity* collineation of a projective plane if it fixes every point and every line of the projective plane.

NOTE. The notation for collineations described above occasionally causes confusion when more than one collineation is being used. In particular, it should be noted that if γ and δ are both collineations of some projective plane, then $P^{\gamma\delta}$ means $(P^\gamma)^\delta = \delta(\gamma(P))$. That is, $\gamma\delta$ is $\delta \circ \gamma$. This change in the order of γ and δ between the "exponential product" and "composition of functions" notations can be hard to keep straight.

One can, of course, extend the notion of collineation to affine planes and other incidence structures.

EXAMPLE 3.1. We can define a collineation α of $PG(2, \mathbb{R})$ which is not the identity as follows in terms of homogeneous coordinates.

- For each point (x, y, z) of $PG(2, \mathbb{R})$, let $(x, y, z)^\alpha = (y, x, z)$, and
- for each line $[a, b, c]$ of $PG(2, \mathbb{R})$, let $[a, b, c]^\alpha = [b, a, c]$.

It is pretty easy to check that α preserves incidence. Note that α fixes precisely those points and lines which have homogeneous coordinates of the forms (x, x, z) and $[a, a, c]$ respectively.

EXAMPLE 3.2. We can define another collineation, β, of $PG(2,\mathbb{R})$ which is not the identity as follows in terms of affine coordinates (see Definition 2.2). Let $c \in \mathbb{R}$ be a constant.

- For each point (x, y) of $PG(2,\mathbb{R})$, let $(x, y)^\beta = (x + c, y)$,
- for each point (m) of $PG(2,\mathbb{R})$, let $(m)^\beta = (m)$, and
- for the point (∞) of $PG(2,\mathbb{R})$, let $(\infty)^\beta = (\infty)$.

Note that every point on the line at infinity is fixed by β, and hence so is the line itself.

PROBLEM 3.1. *Work out how all the other lines of $PG(2,\mathbb{R})$ are moved by the collineation β defined in Example 3.2.*

PROBLEM 3.2. *Work out how, in terms of homogeneous coordinates, points and lines of $PG(2,\mathbb{R})$ are moved by the collineation β defined in Example 3.2.*

EXAMPLE 3.3. Suppose F is a field and $\mathbf{A} \in M_3(F)$ is an invertible 3×3 matrix over F. We can define a collineation δ of $PG(2, F)$, in terms of homogeneous coordinates, by letting $(x, y, z)^\delta = \left(\mathbf{A}(x, y, z)^T\right)^T$.

PROBLEM 3.3. *Verify that the δ of Example 3.3 is indeed a collineation of $PG(2, F)$.*

Some properties of collineations. First, a trivial observation:

PROPOSITION 3.4. *The identity collineation of a projective plane is indeed a collineation of that plane.*

It will sometimes be convenient to know just how much of a projective plane must be fixed by a given collineation before we can conclude it is actually the identity collineation. Here are a couple of shortcuts:

PROPOSITION 3.5. *Suppose $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane, ℓ is a line of $\mathcal{L}$, and Q and R are points of $\mathcal{P}$ which are not incident with ℓ. If γ is a collineation of π that fixes Q, R, and every point P on ℓ, then γ is the identity.*

PROPOSITION 3.6. *Suppose $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane and γ is a collineation of π that fixes all the points of a quadrangle. Then γ fixes the subplane generated by the quadrangle.*

It is important to know that a composition of collineations gives another collineation...

PROPOSITION 3.7. *If γ and δ are collineations of some projective plane, then $\gamma\delta$ (*i.e.* $\delta \circ \gamma$) is also a collineation of the plane.*

...and that the inverse function of a collineation is also a collineation...

PROPOSITION 3.8. *If γ is a collineation of some projective plane, then γ^{-1} is also a collineation of the plane.*

...because it follows that:

PROPOSITION 3.9. *The set of all the collineations of a projective plane forms a group under composition.*

Unfortunately, even a small plane may have a pretty large collineation group.

PROBLEM 3.10. *Find all the collineations of the Fano configuration.*[1]

With some effort, one can use free completion to construct planes with only the identity as a collineation.

We will study how groups of collineations act on the plane, *i.e.* how the collineations in these groups fix and/or move points and lines, in the next chapter.

Axial and central collineations. The most important collineations for our purposes will be those that fix all the points on a line or all the lines through a point.

DEFINITION 3.2. A collineation γ of a projective plane is

(1) an *axial collineation* if it fixes all points on some line ℓ (the *axis*), *i.e.* $P^\gamma = P$ for every point P such that $PI\ell$, and is
(2) a *central collineation* if it fixes all lines through of some point P (the *centre*), *i.e.* $\ell^\gamma = \ell$ for every line ℓ such that $PI\ell$.

PROBLEM 3.11. *Since the collineation β of $PG(2,\mathbb{R})$ given in Example 3.2 fixes every point on the line at infinity, it is axial. Show that it is also central.*

EXAMPLE 3.4. We can define a collineation γ of $PG(2,\mathbb{R})$ which is neither axial nor central as follows in terms of homogeneous coordinates.

- For each point (x, y, z) of $PG(2,\mathbb{R})$, let $(x, y, z)^\gamma = (z, x, y)$, and
- for each line $[a, b, c]$ of $PG(2,\mathbb{R})$, let $[a, b, c]^\gamma = [c, a, b]$.

As in Example3.1, it is pretty easy to check that γ preserves incidence. Note that γ fixes only the point with homogeneous coordinates $(1, 1, 1)$ and the line with homogeneous coordinates $[1, 1, 1]$. Since every other point and line is moved by γ, it is neither an axial nor a central collineation.

[1] If you feel *really* ambitious – or masochistic – you can also work out the group table of the collineation group of the Fano configuration.

PROBLEM 3.12. *Is the collineation α of $PG(2, \mathbb{R})$ given in Example 3.1 central or axial?*

PROBLEM 3.13. *Find all the axial and central collineations of the Fano configuration.*

A key fact about axial and central collineations is that the two definitions actually coincide.

THEOREM 3.14. *A collineation γ of a projective plane is an axial collineation if and only if it is a central collineation.*

This justifies the first part of the following definition.

DEFINITION 3.3. A central collineation with centre P and axis ℓ is often referred to as a (P, ℓ)-*central collineation* or a (P, ℓ)-*collineation*. It is said to be

(1) an *elation* if $PI\ell$, and
(2) a *homology* if $P \not{I} \ell$.

For example, the collineation given in Example 3.2 is an elation, as you should have found that its centre is a point on the line at infinity in Problem 3.11.

PROBLEM 3.15. *Find a homology of $PG(2, \mathbb{R})$.*

Some properties of (P, ℓ)-central collineations. A (P, ℓ)-central collineation fixes P and ℓ, as well as every line through P and every point on ℓ, by definition, but (unless it is the identity) it will move other points and lines, albeit with some restrictions. The following lemma gives those restrictions and can also serve as an useful tool in locating the centre and/or the axis.

LEMMA 3.16. *Suppose γ is a (P, ℓ)-central collineation. If Q is a point that is not fixed by γ, then Q and Q^γ are collinear with P. Similarly, if m is a line that is not fixed by γ, then m and m^γ are coincident with ℓ.*

It will be useful later on to know that for a given point P and line ℓ, the (P, ℓ)-central collineations form a group under composition.

PROPOSITION 3.17. *If γ and δ are (P, ℓ)-central collineations, then $\gamma\delta$ (*i.e. *$\delta \circ \gamma$) is also a (P, ℓ)-central collineation.*

PROPOSITION 3.18. *If γ is a (P, ℓ)-central collineation, then γ^{-1} is also a (P, ℓ)-central collineation.*

THEOREM 3.19. *If $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane, G is any group of collineations of π, $P \in \mathcal{P}$, and $\ell \in \mathcal{L}$, then the (P, ℓ)-central collineations of π which are in G form a subgroup of G. (This subgroup of G is usually denoted by $G(P, \ell)$.)*

PROBLEM 3.20. *Is a product of two central collineations, not necessarily with the same centre and axis, necessarily a central colineation too?*

We will develop further properties of central collineations in the next chapter.

CHAPTER 4

Transitivity and Desargues' Theorem

In this chapter we will establish a connection between the central collineations of a plane and its geometrical properties.

Transitivity. We will be particularly interested if the (P, ℓ)-central collineations, taken collectively, move points about with no further restrictions than those imposed by Lemma 3.16.

DEFINITION 4.1. Suppose $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane, $P \in \mathcal{P}$, and $\ell \in \mathcal{L}$. π is said to be *(P, ℓ)-transitive* if, given any two points A and B which are collinear with but different from P and which are not on ℓ, there is a (P, ℓ)-central collineation γ of π such that $A^\gamma = B$.

EXAMPLE 4.1. $PG(2, \mathbb{R})$, equipped with extended affine coordinates, is $((0), [\infty])$-transitive. If (a, b) is a point of $PG(2, \mathbb{R})$ not on $[\infty]$, then any other point on the line joining it to (0) (other than (0) itself) has coordinates (z, b) for some z. Let (e, b) be such a point. The $((0), [\infty])$ -central collineation given by $(x, y)^\gamma = (x + e - a, y)$ moves (a, b) to $(a + e - a, b) = (e, b)$ (see Example 3.2 and set $c = e - a$).

PROBLEM 4.1. *Show that $PG(2, \mathbb{R})$ is also $((0, 0), [\infty])$-transitive.*

The following proposition is often useful in checking that a plane is transitive at many point-line pairs.

PROPOSITION 4.2. *Suppose $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane which is (P, ℓ)-transitive for some point $P \in \mathcal{P}$ and line $\ell \in \mathcal{L}$, and suppose α is any collineation of π. Then π is also (P^α, ℓ^α)-transitive.*

PROBLEM 4.3. *Show that the Fano configuration is (P, ℓ)-transitive for every point-line pair (P, ℓ).*

In fact, every plane coordinatized by a skew field is transitive for every point-line pair.

LEMMA 4.4. *Suppose F is a skew field. Then $PG(2, F)$ is $((0), [\infty])$- and $((0, 0), [\infty])$-transitive.*

LEMMA 4.5. *Suppose F is a skew field, and P and Q are points and ℓ and m are lines of $PG(2, F)$ such that either $PI\ell$ & QIm or $P \not{I} \ell$ &*

$Q \not I m$. Then there is a collineation α of $PG(2,F)$ such that $P^\alpha = Q$ and $\ell^\alpha = m$.

PROPOSITION 4.6. *Suppose F is a skew field. Then $PG(2,F)$ is (P,ℓ)-transitive for every point-line pair (P,ℓ).*

Not every projective plane is transitive at every point-line pair. For example, as we shall see below, the projective version of the Moulton plane described in Example 2.1 is not transitive for some point-line pair (P,ℓ). We will also use free completion to construct a plane which is not (P,ℓ)-transitive for any point-line pair (P,ℓ) whatsoever.

Desargues' Theorem. It turns out that a plane is (P,ℓ)-transitive exactly when a suitable geometrical condition holds. You may have seen some variation of the following result:

THEOREM 4.7 (Desargues' Theorem). *Suppose ABC and DEF are triangles in a projective plane. Then they are in perspective from a point,* i.e. *the lines AD, BE, and CF intersect in a common point, if and only if they are in perspective from a line,* i.e. *the points $AB \cap DE$, $AC \cap DF$, and $BC \cap EF$ are collinear.*

A restricted version of Desargues' Theorem turns out to be equivalent to (P,ℓ)-transitivity.

DEFINITION 4.2. Suppose $\pi = (\mathcal{P},\mathcal{L},I)$ is a projective plane, $P \in \mathcal{P}$, and $\ell \in \mathcal{L}$. π is (P,ℓ)-*Desarguesian* if whenever any two triangles ABC and DEF of π are in perspective from P, *i.e.* AD, BE, and CF all intersect in P, and both $AB \cap DE$ and $BC \cap EF$ are on ℓ, then $AC \cap DF$ is also on ℓ.

That is, π is (P,ℓ)-Desarguesian if whenever two triangles are in perspective from P and ought also to be in perspective from ℓ, they really are in perspective from ℓ. See Figure 1 for a diagram of the configuration involved.

LEMMA 4.8. *Desargues' Theorem holds in a projective plane $\pi = (\mathcal{P},\mathcal{L},I)$ if and only if π is (P,ℓ)-Desarguesian for every point $P \in \mathcal{P}$ and line $\ell \in \mathcal{L}$.*

THEOREM 4.9. *Suppose $\pi = (\mathcal{P},\mathcal{L},I)$ is a projective plane, $P \in \mathcal{P}$, and $\ell \in \mathcal{L}$. Then π is (P,ℓ)-transitive if and only if π is (P,ℓ)-Desarguesian.*

One consequence of this is to make it easier to check when transitivity fails, since it is usually far easier to find a specific failure of Desargues' Theorem than to check that no (P,ℓ)-collineation will move some point to some other point.

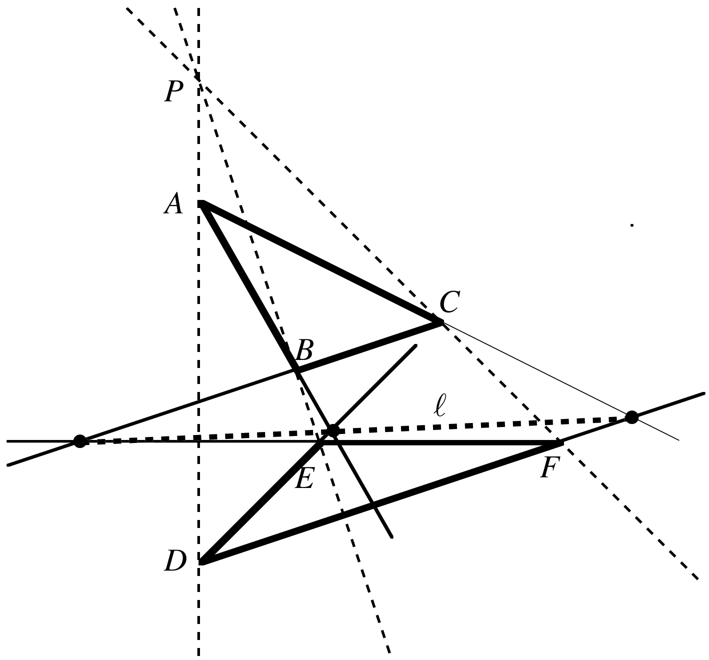

FIGURE 1. Desargues' configuration

PROBLEM 4.10. *Show that the Moulton plane is not* (P, ℓ)*-Desarguesian (and hence is not* (P, ℓ)*-transitive) for some point-line pair* (P, ℓ)*.*

PROBLEM 4.11. *Show that the free completion of a quadrangle is not* (P, ℓ)*-Desarguesian, and hence is not* (P, ℓ)*-transitive, for every point-line pair* (P, ℓ)*.*

CHAPTER 5

Coordinatization

In this chapter we will show how, given nothing but a projective plane as an incidence structure, one can introduce a coordinate system in a projective plane and define a suitable algebraic structure for the coordinates. We will then establish various relationships between the geometric properties of the plane and the algebraic properties of its coordinate system.

Introducing coordinates. There are several methods available to construct a coordinate system for a projective plane, starting only with its incidence structure. They differ in their details, but ultimately give the same principal results. The method we will use is due to G. Pickert and constructs a system of extended affine coordinates that comes as near as possible in general to making the lines (which ought to be) of the form $y = mx + b$ work in familiar ways.

The basic idea is to start with some set of symbols R (including 0 and 1) to be used as coordinates, pick a quadrangle of points that will be assigned the coordinates $(0, 0)$, $(1, 1)$, (0), and (∞), and then assign the rest of the symbols in some arbitrary way to the points on the line which will have the equation $y = x$. One can then use the incidence structure of the plane, together the usual interaction of "horizontal" and "vertical" lines with affine cordinates, to determine the coordinates of all affine points, after which finding the coordinates of the remaining points on the line at infinity is pretty easy. Finally, once one has the coordinates of all the points nailed down, nailing down the coordinates of all the lines is pretty easy.

Definition 5.1. Suppose that $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane and that R is a set of symbols, including 0 and 1, which is just large enough to assign a symbol from R to each point of some line in the affine plane corresponding to π. (That is, $|\mathcal{P}| = |R| + 1$.)

Choose a quadrangle $OEUV$ in π (the *fundamental quadrangle* of the coordinate system) and declare their coordinates to be $O = (0, 0)$, $E = (1, 1)$, $U = (0)$, and $V = (\infty)$, as in Figure 1. Give every other point on the line OE coordinates of the form (a, a) for some distinct a in $R \setminus \{0, 1\}$.

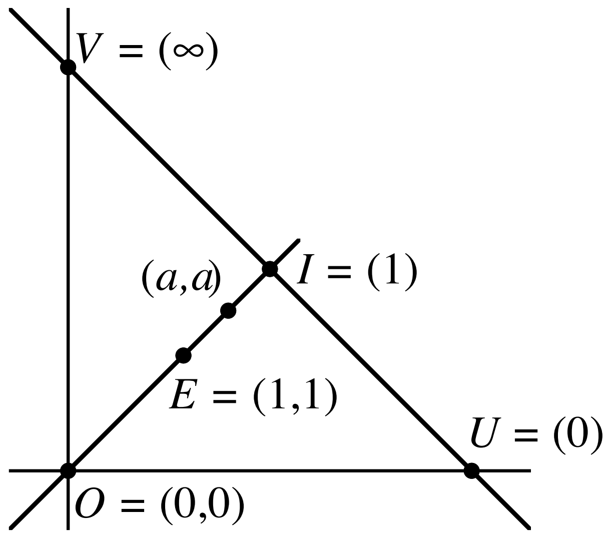

FIGURE 1. The fundamental quadrangle

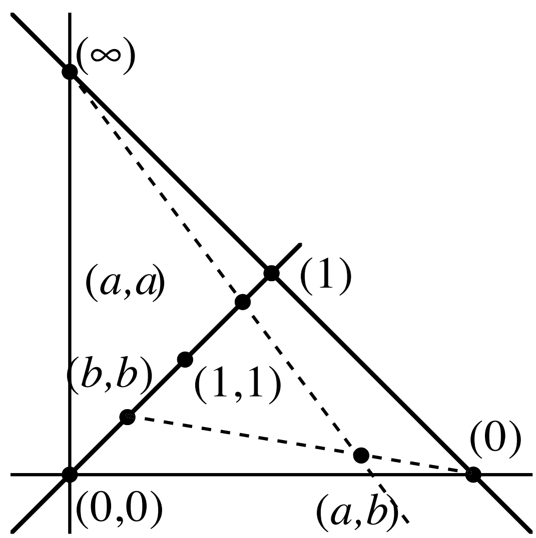

FIGURE 2. Coordinates of affine points

For any point X not incident with OE or UV, we can assign coordinates by setting $X = (a, b)$ if $XV \cap OE = (a, a)$ and $XU \cap OE = (b, b)$, as in Figure 2.

Finally, give each point Y on UV, other than U or V, coordinates by setting $Y = (m)$ if $OY \cap EV = (1, m)$, as in Figure 3.

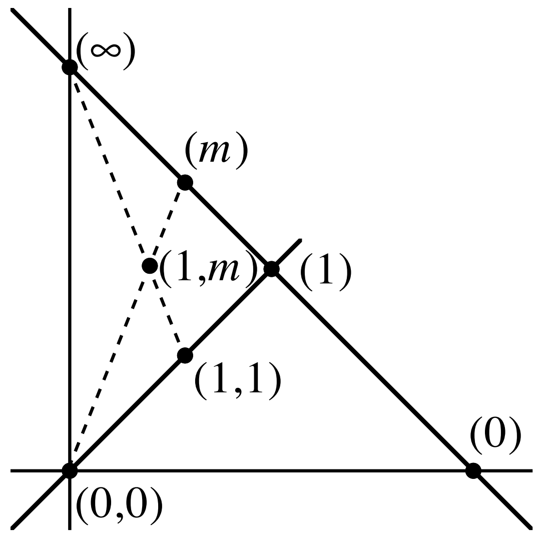

FIGURE 3. Coordinates of ideal points

PROBLEM 5.1. *Complete the definition of how coordinates are introduced by working out what the coordinates of the various lines ought to be, at least if we wish to have affine coordinates work as usual.*

NOTE. On a point of notation, recall that the line (which ought to be) described by the equation $y = mx + b$ is often denoted by $[m, k]$, the line $x = c$ by $[c]$, and the line at infinity by $[\infty]$. As we shall see below, we will not always be able to define operations of $+$ and $\cdot$ on our coordinates so as to ensure that the set of points (x, y) satisfying $y = mx + b$ is always a line.

PROBLEM 5.2. *Begin with the Fano configuration, choose a fundamental quadrangle, and work out the coordinates of all the points and lines. How do these compare with those obtained in the construction the Fano configuration from the two-element field $F = GF(2)$ using extended affine coordinates?*

Ternary rings. The algebraic structure which we can be sure of getting when we introduce coordinates in a projective plane is not usually as nice as a field or skew field. We can only guarantee, in particular, that we get a structure in which the key operation is ternary (*i.e.* three-place).

DEFINITION 5.2. If coordinates (using symbols from a set R) are introduced in a projective plane π as above, the corresponding ternary operation $T : R^3 \to R$ is defined by

$$y = T(m, x, b) \iff (x, y)I[m, k] .$$

In the familiar case that π is defined using a (skew) field, the lines are given by $y = mx + b$, so the corresponding ternary operation is $T(m, x, b) = mx + b$. In general, though, what we can guarantee in general is summed up in the following theorem.

THEOREM 5.3. *If coordinates (using symbols from a set R) are introduced in a projective plane π as above, then the corresponding ternary operation $T : R^3 \to R$ satisfies the following conditions:*

(1) *For all $x, b \in R$, $T(x, 0, b) = T(0, x, b) = b$.*
(2) *For all $x \in R$, $T(1, x, 0) = T(x, 1, 0) = x$.*
(3) *For all $x, y, u, v \in R$ with $x \neq u$, there is an unique ordered pair $(m, b) \in R^2$ such that $y = T(m, x, b)$ and $v = T(m, u, b)$.*
(4) *For all $x, y, m \in R$, there is an unique $b \in R$ with $y = T(m, x, b)$.*
(5) *For all $m, b, n, c \in R$ with $m \neq n$, there is an unique $x \in R$ with $T(m, x, b) = T(n, x, c)$.*

DEFINITION 5.3. A *planar ternary ring* (occasionally called a *planar ternary field*) is a set R, including the symbols 0 and 1, together with a ternary operation $T : R^3 \to R$ satisfying the conditions 1–5 of Theorem 5.3.

Given a planar ternary ring, we can construct the corresponding projective plane quite readily.

THEOREM 5.4. *Suppose (R, T) is a planar ternary ring. Define an incidence structure $\pi = (\mathcal{P}, \mathcal{L}, I)$ as follows:*

- $\mathcal{P} = \{\, (x, y) \mid x, y \in R \,\} \cup \{\, (m) \mid m \in R \,\} \cup \{(\infty)\}$
- $\mathcal{L} = \{\, [m, b] \mid m, b \in R \,\} \cup \{\, [c] \mid c \in R \,\} \cup \{[\infty]\}$
- *Incidence is defined as follows:*
  - $(x, y) I [m, b] \iff y = T(m, x, b)$
  - $(x, y) I [c] \iff x = c$
  - $(x, y) \not{I} [\infty]$
  - $(n) I [m, b] \iff n = m$
  - $(n) \not{I} [c]$
  - $(n) I [\infty]$
  - $(\infty) \not{I} [m, b]$
  - $(\infty) I [c]$
  - $(\infty) I [\infty]$

Then π is a projective plane.

Some basic properties of ternary rings. We can use a planar ternary ring (R, T) to define operations of addition and multiplication on R, though, unfortunately, these will not necessarily be terribly well-behaved.

DEFINITION 5.4. Suppose (R, T) is a planar ternary ring. $+$ and $\cdot$ on R are defined by setting, for $a, b \in R$, $a + b = T(1, a, b)$ and $a \cdot b = T(a, b, 0)$

DEFINITION 5.5. A non-empty set G with a binary operation $\circ$ is a *loop* if it satisfies the following conditions.

- For all $a, b \in G$, there is an unique $x \in G$ such that $a \circ x = b$.
- For all $a, b \in G$, there is an unique $y \in G$ such that $y \circ a = b$.
- G has an identity element e for $\circ$, *i.e.* $x \circ e = e \circ x = x$ for all $x \in G$.

PROPOSITION 5.5. *If (R, T) is a planar ternary ring, then $(R, +)$ and $(R \setminus \{0\}, \cdot)$ are loops.*

Unfortunately, this proposition is about as much as we can say without additional assumptions about the ternary ring or the plane it is defined from. Indeed, we cannot even assume that $T(m, x, b) = mx + b$ without additional assumptions.

PROPOSITION 5.6. *Suppose π is the free completion of a quadrangle and (R, T) is the planar ternary ring defined from π using the original quadrangle as the fundamental quadrangle. Then*

- *$T(m, x, b) \neq mx + b$ for some $m, x, b \in R$.*
- *$+$ is not associative.*
- *$\cdot$ is not associative.*
- *$+$ is not commutative.*
- *$\cdot$ is not commutative.*
- *The distributive laws for $+$ and $\cdot$ fail.*

It follows from the first item in the proposition above that not every line that ought to be is of the form $y = mx + b$; it follows from the other items that $(R, +)$ and $(R, \cdot)$ are not groups, and that $(R, +, \cdot)$ is not a ring, much less a skew field or a field.

CHAPTER 6

Geometric *vs.* algebraic properties

The geometric properties of a projective plane and the algebraic properties of the ternary ring coordinatizing it are closely tied together. A few of the most basic connections are established below. In what follows, we will suppose that $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane and (R, T) is the ternary ring coordinatizing π with respect to the fundamental quadrangle $OEUV$.

First, however, a small fact from algebra:

LEMMA 6.1. *Suppose $(G, \circ)$ is a loop and the binary operation $\circ$ is associative. Then $(G, \circ)$ is a group.*

Linearity. One nice property that we would like to have a ternary ring (R, T) satisfy is that addition and multiplication behave well enough to ensure that $T(m, x, b) = mx + b$.

DEFINITION 6.1. A ternary ring (R, T) is *linear* if $T(m, x, b) = mx + b$ for all $m, x, b \in R$.

It turns out that (R, T) is linear exactly when the projective plane it coordinatizes satisfies a highly restricted version of Desargues' Theorem, even more restricted than the (P, ℓ)-Desargues' Theorem.

PROPOSITION 6.2. *(R, T) is linear if and only if whenever ABC and DEF are triangles of π which are*

(1) *in perspective from V so that*
(2) *A and D are incident with OV,*
(3) *$AB \cap DE$ and $AC \cap DF$ are incident with UV, and*
(4) *BC is incident with U,*

it also the case that EF is incident with U.

It follows that very modest amounts of transitivity in π with respect to point-line pairs in the fundamental quadrangle suffice to ensure that (R, T) is linear.

COROLLARY 6.3. *If π is (V, UV)-transitive, then (R, T) is linear.*

PROPOSITION 6.4. *If π is (U, OV)-transitive, then (R, T) is linear.*

Additive properties. A modest amount of transitivity will also ensure that addition is fairly well-behaved.

DEFINITION 6.2. A ternary ring (R, T) is a *Cartesian group* if it is linear and $(R, +)$ is associative.

It follows from Lemma 6.1 that if (R, T) is a Cartesian group, then $(R, +)$ is a group.

THEOREM 6.5. *(R, T) is a Cartesian group if and only if π is (V, UV)-transitive.*

A little more transitivity will also get addition to interact even better with multiplication.

DEFINITION 6.3. A ternary ring (R, T) satisfies the *left distributive law* if $a(b + c) = ab + ac$ for all $a, b, c \in R$.

THEOREM 6.6. *(R, T) is a Cartesian group satisfying the left distributive law if and only if π is (V, UV)-transitive and (U, UV)-transitive.*

Multiplicative properties. The results above relating transitivity in the fundamental quadrangle to properties of addition have close counterparts relating transitivity in the fundamental quadrangle to properties of multiplication.

THEOREM 6.7. *(R, T) is linear and $(R, \cdot)$ is associative if and only if π is (U, OV)-transitive.*

It follows from Lemma 6.1 that if (R, T) has associative multiplication, then $(R \setminus \{0\}, \cdot)$ is a group. Again, a little more transitivity will get multiplication to interact better with addition.

THEOREM 6.8. *(R, T) is linear with associative multiplication and satisfies the left distributive law if and only if π is (U, OV)-transitive and (V, OU)-transitive.*

PROBLEM 6.9. *How little transitivity relative to the fundamental quadrangle $OEUV$ is required to ensure that the resulting ternary ring is at least a skew field? How much more transitivity follows in such a case?*

APPENDIX A

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Version 1.2, November 2002

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