Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry
Trent University, Fall 2019

Solutions to Assignment #4
Collineations

Recall from class that a *collineation* of a projective plane is a 1-1 onto function α that takes points of the projective plane to points of the projective plane, lines of the projective plane to lines of the projective plane, and preserves incidence. *i.e.* $P \mathbf{I} \ell \Leftrightarrow P^\alpha \mathbf{I} \ell^\alpha$. It is traditional to write P^α for $\alpha(P)$ and similarly for lines; compositions of collineations work "inside first" in this notation, *e.g.*. $P^{\alpha\beta} = P^{\beta\circ\alpha} = \beta(\alpha(P))$.

1. Show that if α is a collineation of a projective plane, then α^{-1} is also a collineation of the same projective plane. *[2]*

Solution. Since α takes points to points and lines to lines, α^{-1} must as well. Moreover, since α is 1-1 and onto, α^{-1} is as well. It remains to check that α^{-1} preserves incidence. Since α preserves incidence, we have

$$P^{\alpha^{-1}} \mathbf{I} \ell^{\alpha^{-1}} \iff P^{\alpha^{-1}\alpha} \mathbf{I} \ell^{\alpha^{-1}\alpha} \iff P \mathbf{I} \ell$$

for any point P and line ℓ, so α^{-1} preserves incidence. Thus α^{-1} is a collineation if α is a collineation. $\square$

2. Show that if α and β are collineations of a projective plane, then $\alpha\beta = \beta\circ\alpha$ is also a collineation of the same projective plane. *[2]*

Solution. Since α and β take points to points and lines to lines, $\alpha\beta = \beta\circ\alpha$ does as well. Moreover, since both α and β are 1-1 and onto, so is $\alpha\beta = \beta\circ\alpha$. It remains to check that $\alpha\beta = \beta\circ\alpha$ preserves incidence. Since each of α and β preserve incidence, we have

$$P^{\alpha\beta} \mathbf{I} \ell^{\alpha\beta} \iff P^{\alpha} \mathbf{I} \ell^{\alpha} \iff P \mathbf{I} \ell$$

for any point P and line ℓ, so $\alpha\beta = \beta\circ\alpha$ preserves incidence. Hence $\alpha\beta = \beta\circ\alpha$ is a collineation if α and β are collineations. $\square$

Note. For those of you taking abstract algebra, the two problems above do most of the work in showing that the collineations of a projective plane form a group, with the group operation being composition.

Recall also that a collineation is said to be (P,ℓ)-*central* (sometimes referred to as a (P,ℓ)-*perspectivity*) if it has has centre P and axis ℓ, *i.e.* $Q^\alpha = Q$ for every point Q on the axis ℓ and $m^\alpha = m$ for every line m passing through the centre P. (We showed in class that a collineation has an axis if and only if it has a centre.) A (P,ℓ)-central collineation is said to be an *elation* if P is on ℓ, and is said to be a *homology* if P is not on ℓ.

Definition. Two triangles ABC and DEF are said to be *in perspective from a point* P if AD, BE, and CF are all incident with P, and *in perspective from a line* ℓ if $AB \cap DE$, $BC \cap EF$, and $AE \cap DF$ are all incident with ℓ.

3. Suppose α is a (P,ℓ)-central collineation of a projective plane and ABC is a triangle of the projective plane such that none of A, B, and C are P or incident with ℓ. Show that ABC and $A^\alpha B^\alpha C^\alpha$ are in perspective from P and in perspective from ℓ. *[3]*

SOLUTION. Suppose α is a (P,ℓ)-central collineation of a projective plane and ABC is a triangle of the projective plane such that none of A, B, and C are P or incident with ℓ.

Let $D = PA \cap \ell$, $E = PB \cap \ell$, and $F = PC \cap \ell$; then $D^\alpha = D$, $E^\alpha = E$, and $F^\alpha = F$ because all three points are on the axis ℓ. Since P is the centre of the collineation α, we also have $P^\alpha = P$, and so the lines $(PD)^\alpha = P^\alpha D^\alpha = PD$, $(PE)^\alpha = P^\alpha E^\alpha = PE$, and $(PD)^\alpha = P^\alpha F^\alpha = PF$ are fixed by α. It follows that

$$PA^\alpha = P^\alpha A^\alpha = (PA)^\alpha = (PD)\alpha = PD = PA\,,$$
$$PB^\alpha = P^\alpha B^\alpha = (PB)^\alpha = (PD)\alpha = PD = PB\,, \text{ and}$$
$$PC^\alpha = P^\alpha C^\alpha = (PC)^\alpha = (PD)\alpha = PD = PC\,,$$

so A^α is on PA, B^α is on PB, and C^α is on PC, *i.e.* ABC and $A^\alpha B^\alpha C^\alpha$ are in perspective from P.

Let $T = AB \cap \ell$, $U = AC \cap \ell$, and $V = BC \cap \ell$; then $T^\alpha = T$, $U^\alpha = U$, and $V^\alpha = V$ because all three points are on the axis ℓ. It follows that

$$A^\alpha B^\alpha \cap \ell = (AB)\alpha \cap \ell^\alpha = (AT)^\alpha \cap \ell = A^\alpha T^\alpha \cap \ell = A^\alpha T \cap \ell = T\,,$$
$$A^\alpha C^\alpha \cap \ell = (AC)\alpha \cap \ell^\alpha = (AU)^\alpha \cap \ell = A^\alpha U^\alpha \cap \ell = A^\alpha U \cap \ell = U\,, \text{ and}$$
$$A^\alpha B^\alpha \cap \ell = (AB)\alpha \cap \ell^\alpha = (AV)^\alpha \cap \ell = A^\alpha V^\alpha \cap \ell = A^\alpha V \cap \ell = T\,,$$

so $AB \cap A^\alpha B^\alpha = T$, $AC \cap A^\alpha C^\alpha = U$, and $BC \cap B^\alpha C^\alpha = V$, which are all on the axis ℓ, *i.e.* ABC and $A^\alpha B^\alpha C^\alpha$ are in perspective from ℓ. □

4. Suppose α is a (P,ℓ)-central collineation of a projective plane and A and B are points of the projective plane which are not on ℓ and such that A, B, and P are not collinear, and that we know A^α and B^α. Show that this completely determines α, *i.e.* given any point C in the plane, show how to find C^α, and given any line m, show how to find m^α. *[3]*

SOLUTION. Knowing B^α is a bit of a red herring here ...

Suppose α is a (P,ℓ)-central collineation of a projective plane, $A \neq P$ is a point not on ℓ for which we know A^α, and C is any point not on PA or ℓ. Note that $(PA)^\alpha = PA$ and $(PC)^\alpha = PC$ because all lines through the centre are fixed by a (P,ℓ)-central collineation, and if we let $D = AC \cap \ell$, then $D^\alpha = D$ because all points on the axis ℓ are fixed by a (P,ℓ)-central collineation.

C^α must be on PC because $PC = (PC)^\alpha = P^\alpha C^\alpha PC^\alpha$, and C^α must be on $A^\alpha D = A^\alpha D^\alpha$ because C is on AD and α preserves incidence. It follows that $C^\alpha = PC \cap A^\alpha D$. This means that α is completely determined for all points not on PA or ℓ once we know A^α.

Since the centre and the points on the axis are fixed by any (P,ℓ)-central collineation, it only remains to figure out how α moves points on PA that are not P, A, or $PA \cap \ell$. If we pick such a point X on PA and any point C not on PA or ℓ, we can pin down X^α by the same reasoning used in the previous paragraph, with the role played there by A now played by C and the role played there by C now played by X.

Once we know how α moves all the points, finding out how it moves lines is trivial: if m is any line, pick two points Q and R on m, and then $m^\alpha = (QR)^\alpha = Q^\alpha R^\alpha$. ■