Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry
Trent University, Fall 2019

Solutions to Assignment #3
Collineations of the Real Projective Plane from Linear Algebra

Recall from class that we can, among other ways, define the real projective plane using *projective coordinates*:

- Points are represented by non-zero vectors $(a, b, c) \in \mathbb{R}^3$, and another vector (d, e, f) represents the same point if there is a scalar $\lambda \neq 0$ such that $(a, b, c) = \lambda(d, e, f)$.
- Lines are represented by non-zero vectors $[p, q, r] \in \mathbb{R}^3$, and another vector $[s, t, u]$ represents the same point if there is a scalar $\lambda \neq 0$ such that $[p, q, r] = \lambda[s, t, u]$.
- A point (a, b, c) is incident with a line $[p.q.r]$, often written as $(a, b, c)\,\mathbf{I}\,[p.q.r]$, if and only if $(a, b, c) \cdot [p, q, r] = ap + bq + cr = 0$.

Suppose $\mathbf{M}$ is a 3×3 invertible matrix with real entries. Define a function φ that maps points of the real projective plane to points of the real projective plane by $\varphi(P) = \left(\mathbf{M}P^T\right)^T = P\mathbf{M}^T$. (The transposes are there because points are represented by row vectors and matrix multiplication is commonly defined for column vectors.)

1. Verify that φ does indeed take points of the real projective plane to points of the real projective plane, and is also 1–1 and onto. *[5]*

Solution. By it's definition, φ moves row vectors in $\mathbb{R}^3$ to row vectors in $\mathbb{R}^3$. As a sanity check, note that because $\mathbf{M}$ is invertible, $\mathbf{M}\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$, so φ takes non-zero vectors to non-zero vectors. Now if $P = (a.b.c)$ and $\lambda \neq 0$ is a scalar, then $\varphi(\lambda P) = \left(\mathbf{M}(\lambda P)^T\right)^T = \lambda\left(\mathbf{M}P^T\right)^T$ because multiplication by scalars passes through matrix multiplication and taking transposes. Thus φ takes different projective coordinates representing the same input point to different projective coordinates for the same output point, so it properly takes points to points.

Since $\mathbf{M}$ is an invertible matrix, φ is an invertible function taking points to points: if we let $\varphi^{-1}(P) = \left(\mathbf{M}^{-1}P^T\right)^T$, then

$$\varphi^{-1}\left(\varphi(P)\right) = \left(\mathbf{M}^{-1}\left(\left(\mathbf{M}P^T\right)^T\right)^T\right)^T = \left(\mathbf{M}^{-1}\mathbf{M}P^T\right)^T = \left(P^T\right)^T = P\,, \text{ and}$$

$$\varphi\left(\varphi^{-1}(P)\right) = \left(\mathbf{M}\left(\left(\mathbf{M}^{-1}P^T\right)^T\right)^T\right)^T = \left(\mathbf{M}\mathbf{M}^{-1}P^T\right)^T = \left(P^T\right)^T = P\,.$$

Any invertible function must be $1 - 1$ and onto. □

2. Find a way to define φ on the lines so that is a $1 - 1$ onto function that takes lines of the real projective plane to lines of the real projective plane and also preserves incidence, *i.e.* has $\varphi(P)\,\mathbf{I}\,\varphi(\ell) \Leftrightarrow P\,\mathbf{I}\,\ell$ for all point P and lines ℓ of the real projective plane. Verify that your definition does the job! *[5]*

Solution. Since the projective coordinates of lines are defined in much the same way as the projective coordinates of points, we'll try to define φ on the lines in a way similar to the way it is defined for lines, namely $\varphi(\ell) = \left(\mathbf{A}\ell^T\right)^T = \ell\mathbf{A}^T$ for a suitable matrix $\mathbf{A}$. We need to figure out what the matrix $\mathbf{A}$ ought to to ensure that incidence is preserved, *i.e.* $P\,\mathbf{I}\,\ell \Leftrightarrow P \cdot \ell = 0 \Leftrightarrow \varphi(P) \cdot \varphi(\ell) = 0 \Leftrightarrow \varphi(P)\,\mathbf{I}\,\varphi(\ell)$.

Since $\varphi(P) \cdot \varphi(\ell) = \varphi(\ell) \cdot \varphi(P) = \left(\ell\mathbf{A}^T\right) \cdot \left(P\mathbf{M}^T\right) = \left(\ell\mathbf{A}^T\right)\left(P\mathbf{M}^T\right)^T = \ell\mathbf{A}^T\mathbf{M}P^T$, making $\mathbf{A}^T = \mathbf{M}^{-1}$, so $\mathbf{A} = \left(\mathbf{M}^{-1}\right)^T$ ought to work. Let's check this:

$$\varphi(P) \cdot \varphi(\ell) = \ell\mathbf{A}^T\mathbf{M}P^T = \ell\mathbf{M}^{-1}\mathbf{M}P^T = \ell P^T = \ell \cdot P = P \cdot \ell$$

It follows that $P\,\mathbf{I}\,\ell \Leftrightarrow P \cdot \ell = 0 \Leftrightarrow \varphi(P) \cdot \varphi(\ell) = 0 \Leftrightarrow \varphi(P)\,\mathbf{I}\,\varphi(\ell)$, *i.e.* φ so defined preserves incidence.

Note that if $\mathbf{M}$ is invertible, so is $\mathbf{A} = \left(\mathbf{M}^{-1}\right)^T$. It follows that φ is $1 - 1$ and onto on the lines of the real projective plane, using the same argument as was used for points and $\mathbf{M}$ in solving question **1**. ■