Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry
TRENT UNIVERSITY, Fall 2019

Solution to Assignment #2
The Real Projective Plane via Linear Algebra

Recall from class that a projective plane is a geometry consisting of a set of points and and a set of lines satisfying the following axioms:

PI. Any two distinct points are connected by an unique line.
PII. Any two distinct lines intersect in an unique point.
PIII. There exist four points such that no three are on the same line.

The following is one of several equivalent methods of defining the real projective plane.

- The points of the plane are the one-dimensional subspaces of $\mathbb{R}^3$.
- The lines of the plane are the two-dimensional subspaces of $\mathbb{R}^3$.
- A point of the plane is incident with a line of the plane exactly when the one-dimensional subspace is contained in the two-dimensional subspace.

1. Verify that the real projective plane so defined is indeed a projective plane. *[10]*

Hint. You should probably do this by directly showing that this construction satisfies the three axioms for a projective plane. If you really must do it by showing that this way of defining the real projective plane is equivalent to constructing the real projective plane using extended affine coordinates, that is probably best done indirectly, such as by showing that each of the two methods is equivalent to a suitable third method ...

SOLUTION. We'll work through the axioms in order:

PI. Suppose P and Q are two distinct points, *i.e.* they are different one-dimensional subspaces of $\mathbb{R}^3$. Choose non-zero vectors $\mathbf{u} \in P$ and $\mathbf{v} \in Q$; then $P = \text{span}\{\mathbf{u}\}$ and $Q = \text{span}\{\mathbf{v}\}$. Since P and Q are different one-dimensional subspaces of $\mathbb{R}^3$, $\mathbf{u}$ and $\mathbf{v}$ cannot be scalar multiples of one another, *i.e.* they are linearly independent. Let $\ell = \text{span}\,\{\mathbf{u}, \mathbf{v}\}$. Then ℓ is a two-dimensional subspace of $\mathbb{R}^3$ with $P = \text{span}\{\mathbf{u}\} \subset \text{span}\,\{\mathbf{u}, \mathbf{v}\} = \ell$ and $Q = \text{span}\{\mathbf{v}\} \subset \text{span}\,\{\mathbf{u}, \mathbf{v}\} = \ell$, *i.e.* ℓ is a line of the real projective plane which is incident with both the point P and the point Q.

Why is ℓ unique? Any line – *i.e.* two-dimensional subspace – that contains the points – *i.e.* one-dimensional subspaces – P and Q must include the two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ and hence contains their span ℓ. Since this span is two-dimensional, and the only two-dimensional subspace of a two-dimenaional subspace is itself, the two-dimesional subspace – *i.e.* line – in question must be ℓ.

PII. Suppose ℓ and m are two distinct lines, *i.e.* they are different two-dimensional subspaces of $\mathbb{R}^3$. Their intersection $P = \ell \cap m$ must also be a subspace of $\mathbb{R}^3$. It can't be two-dimensional, because then we'd have $\ell = P = m$, contradicting $\ell \neq m$. P can't be zero-dimensional either, because then bases – two vectors each! – for ℓ and m could be united into a set of four linearly independent vectors, which is impossible in a three-dimensional vector space such as $\mathbb{R}^3$. The only remaining possibility is that P has dimension one, *i.e.* P is a point of the real projective plane.

Why is P unique? Well, the intersection of a pair of two-dimensional subspaces is unique ...

PIII. The standard basis vectors of $\mathbb{R}^3$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, together with $\mathbf{u} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, is a set of four vectors of $\mathbb{R}^3$ such that any subset of three of them is a basis for $\mathbb{R}^3$. (This last follows from the fact any one of the three standard basis vectors can be obtained by subtracting the other two standard basis vectors from $\mathbf{u}$.) This means, in particular, that no three of them – and hence the one-dimensional subspaces each spans – can be the same two-dimensional subspace of $\mathbb{R}^3$. But this is exactly what it means for the four points – *i.e.* the four one-dimensional subspaces generated by each of the four vectors – of the real projective plane to have the property that no three are on the same line – *i.e.* be contained in the same two-dimensional subspace of $\mathbb{R}^3$. ■