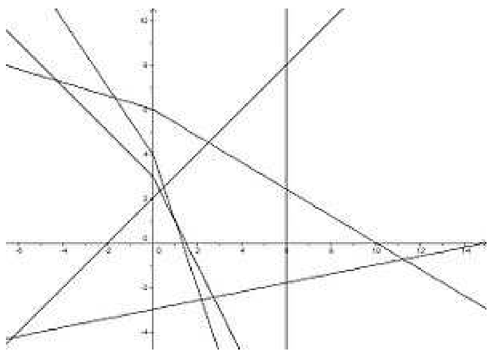
Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry
TRENT UNIVERSITY, Fall 2019

Solution to Assignment #1
A modified Cartesian plane

An affine plane is a geometry consisting of a set of points and and a set of lines satisfying the following axioms:

AI. Any two distinct points are connected by an unique line.
AII. Given a line ℓ and a point P not on ℓ, there is an unique line m through P that has no points in common with ℓ.
AIII. There exist three points that are not all on the same line.

The *Moulton plane* is the affine plane obtained from the Cartesian plane by replacing straight lines with negative slope by lines which bend to double the slope as they cross the y-axis from left to right.



More formally:

- The points of the Moulton plane are the points of the Cartesian plane $\mathbb{R}^2$.
- The lines of the Moulton plane include:
  - The vertical lines of the Cartesian plane, *i.e.* $x = c$ for each $c \in \mathbb{R}$.
  - The lines of non-negative slope of the Cartesian plane, *i.e.* $y = mx + b$ for $m, b \in \mathbb{R}$ with $m \geq 0$.
  - The bent lines given by $y = \begin{cases} mx + b & x \leq 0 \\ 2mx + b & x \geq 0 \end{cases}$ for $m, b \in \mathbb{R}$ with $m \leq 0$.
- A point is on a line of the Moulton plane exactly when its Cartesian coordinates satisfy the equation of the line.

1. Verify that the Moulton plane is indeed an affine plane. *[10]*

SOLUTION. We need to check that each of the three axioms for an affine plane is true of the Moulton plane. In reverse order, with no small amount of overkill:

AIII. Consider the points $(1, 0)$, $(2, 1)$, and $(3, 2)$. The lines joining between two of these points at a time in the Euclidean plane all have positive slope, so they are also lines between them in the Moulton plane. Since the three lines are different from each other, the three points are not on the same line both in the Cartesian plane and the Moulton plane.

AII. Suppose P is a point of the Moulton plane and ℓ is a line of the Moulton plane which is not on ℓ. If ℓ is vertical or has non-negative slope, it is also a line of the Cartesian plane, and the unique line through P in the Cartesian plane which is parallel to ℓ is also the unique line through P in the Moulton plane which is parallel to ℓ. On the other hand, suppose the line ℓ has negative slopes of m on the left side and $2m$ on the right side of the y-axis. Consider the bent line with the same slopes that passes through P.

If P is on the left side of the y axis, the unique line in the Cartesian plane passing through P that is parallel to the left half of ℓ has slope m, and hence there is an unique line which has this slope on the left side of the y axis passing through P in the Moulton plane. This Euclidean line, and hence the Moulton line sharing it's left half, must therefore have a y-intercept different from line ℓ's y-intercept. The corresponding line of the Moulton plane then proceeds to the right of the y-axis from this different point with the same slope $2m$ as ℓ does, so they never intersect on that side either.

Similarly, if P is on the right side of the y-axis, the unique line in the Cartesian plane passing through P that is parallel to the left half of ℓ has slope $2m$, and hence there is an unique line which has this slope on the right side of the y axis passing through P in the Moulton plane. This Euclidean line, and hence the Moulton line sharing it's right half, must therefore have a y-intercept different from line ℓ's y-intercept. The corresponding line of the Moulton plane then proceeds to the right of the y-axis from this different point with the same slope $2m$ as ℓ does, so they never intersect on that side either.

If P is on the y-axis, you could apply either the left-side or right-side arguments above to find the unique line in the Moulton plane through P which is parallel to ℓ.

AI. Suppose P and Q are points of the Moulton plane. If the unique line joining them in the Cartesian plane has non-negative slope or is vertical, then, by the definition of the Moulton plane, it is also the unique line joining the two points in the Moulton plane. Similarly, if P and Q are not on opposite sides of the y-axis, there is an unique Cartesian line joining them on that side of the y-axis. This still gives an unique line of the Moulton plane joining the two points; you just have to bend that unique Cartesian line appropriately when it crosses to y-axis to the side where the two points are not.

The remaining case, where we sadly have to put in some effort, is where one point is to the left of the y axis, say $P = (a, b)$, and the other, say $Q = (c, d)$, is below P and to the right of the y-axis. To show that there is an unique line of the Moulton plane joining these two points it suffices to show that there are unique real numbers $m < 0$ and e such that the Cartesian line joining $P = (a, b)$ to $(0.e)$ has slope m and the Cartesian line joining $(0, e)$ to $Q = (c, d)$ has slope $2m$. Putting the left and right parts, respectively, of these Cartesian lines together will then give us an unique line of the Moulton plane joining P and Q.

Note that we must have $b > e > d$ and $a < 0 < c$ in this situation and that we can compute the slopes of the Cartesian lines using the coordinates of the points they pass through: $m = \dfrac{e-b}{0-a} = \dfrac{b-e}{a}$ and $2m = \dfrac{d-e}{c-0} = \dfrac{d-e}{c}$. It follows that $2 \cdot \dfrac{b-e}{a} = \dfrac{d-e}{c}$, which we can solve for e in terms of a, b, c, and d, namely $e = \dfrac{2bc-ad}{2c-a}$. Plugging this expression for e back into the expression for m lets one solve for m in terms of a, b, c, and d, namely $m = \dfrac{b-d}{a-2c}$. (Calculations left to the reader to save space ... :-) ■