Trent University

Mathematics 3260H – Geometry II: Projective and non-Euclidean Geometry

Take-home Final Examination

Due two weeks from receipt,
or 18 December, whichever is earlier.

Instructions: Give complete answers to receive full credit, including references to any and all sources you used. You may use your texts from this and any other courses, as well as any handouts, class notes, and the like; you may also ask the instructor to clarify the instructions or any of the questions; and you may use a calculator or computer to perform any necessary calculations. *You may not consult any other sources, nor give or receive any other aid on this exam, except with the intructor's explicit permission.*

Part I – Axiom play. Do all *three* (3) of **1** – **3**. *[30 = 3×10 each]*

1. Consider the following geometry $\mathcal{G}$:

i. The points of $\mathcal{G}$ are the points on the cone $z = x^2 + y^2$ in $\mathbb{R}^3$.

ii. The lines of $\mathcal{G}$ are the intersections of the cone with planes through the origin in $\mathbb{R}^3$ that include at least one point of the cone other than the origin.

iii. Distances between points on the cone are measured along the shortest line of $\mathcal{G}$ that connects the two points.

iv. Angles between intersecting lines of $\mathcal{G}$ are the angles between the corresponding planes.

Which of Euclid's Postulates I-IV, along with Playfair's Axiom, are true in $\mathcal{G}$?

2. Suppose $\Pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane and $\ell \in \mathcal{L}$ is one of its lines. Define the incidence structure $\mathcal{A} = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ as follows:

i. $\mathcal{P}^*$ includes every point in $\mathcal{P}$ except those which are on the line ℓ.

ii. $\mathcal{L}^*$ includes every line in $\mathcal{L}$ except ℓ.

iii. I^* is the relation I restricted to the points in $\mathcal{P}^*$ and the lines in $\mathcal{L}^*$.

Show that $\mathcal{A} = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ is an affine plane.

3. The antipodal sphere model of the elliptic plane is also an instance of the real projective plane. If one removes a line from the real projective plane, one is left with the real affine plane, *i.e.* the Euclidean plane. [See **2** above] Thus the Euclidean plane is embedded in a model of the elliptic plane, even though these satisfy very different axioms. Explain why this isn't a problem.

Part II – Projective geometry. Do any *two* (2) of **4** – **6**. *[20 = 2×10 each]*

4. Suppose γ is a collineation of a projective plane which is its own inverse, *i.e.* $\gamma^2 = \gamma \circ \gamma$ is the identity map. Show that γ must fix more than one point and more than one line.

5. Draw (all of) a projective plane which has four points on each line. Determine whether this plane has a (P, ℓ)-central collineation (other than the identity collineation) for some point P and line ℓ such that P is not incident with ℓ.

6. Does the collineation of the real projective plane (using projective coordinates) induced by a 3×3 matrix $\mathbf{M}$ (as in Assignment #3) have to fix some point or not? Either way, prove it.

Part III – Non-Euclidean geometry. Do any *two* (2) of **7** – **10**. *[20 = 2×10 each]*

7. Let ℓ be a line of the elliptic plane, A a point on ℓ, and AB a line segment perpendicular to ℓ. As we slide A along ℓ, keeping $|AB|$ constant, B sweeps out a curve in the elliptic plane. Prove that this curve is not a line of the elliptic plane.

8. Suppose $\triangle ABC$ has $\angle ABC = \angle ACB = \dfrac{\pi}{3}$ and its side BC has length 1. Compute the area of $\triangle ABC$ in ...

- **a.** the Euclidean plane. *[2]*
- **b.** the elliptic plane. *[4]*
- **c.** the hyperbolic plane. *[4]*

9. Is it possible to tile the hyperbolic plane with a triangle? (That is, cover all of the hyperbolic plane with congruent copies of the same triangle, with no overlap between the copies except for the sides and vertices.) Give an example of such a tiling or explain why there can't be one.

[Total = 70]

Part IV - Fun. Bonus!

$\circ^{\circ^{\circ}}$. Write a limerick touching on projective or non-Euclidean geometry in general. *[1]*

I HOPE THAT YOU ENJOYED THE COURSE. ENJOY THE BREAK!