Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry

Trent University, Fall 2019

Assignment #7
Collineations of a free completion

Due on Monday, 4 November.

Suppose $\mathcal{C}_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathbf{I}_0)$ is a configuration of points and lines and an incidence relation between them satisfying:

- For any two points A, $B \in \mathcal{P}_0$, there is at most one (but possibly no) line $\ell \in \mathcal{L}_0$ such that $A\mathbf{I}_0\ell$ and $B\mathbf{I}_0\ell$.
- For any two lines ℓ, $m \in \mathcal{L}_0$, there is at most one (but possibly no) point $P \in \mathcal{P}_0$ such that $P\mathbf{I}_0\ell$ and $P\mathbf{I}_0 m$.

We can then define the *free completion* of $\mathcal{C}_0$ as follows:

For each $n \geq 0$, given a configuration $\mathcal{C}_n = (\mathcal{P}_n, \mathcal{L}_n, \mathbf{I}_n)$, define the configuration $\mathcal{C}_{n+1} = (\mathcal{P}_{n+1}, \mathcal{L}_{n+1}, \mathbf{I}_{n+1})$ as follows:

1. $\mathcal{P}_{n+1}$ includes all the points of $\mathcal{P}_n$, together with a new and distinct point of intersection for every pair of lines in $\mathcal{L}_n$ which do not already have a common point of intersection in $\mathcal{C}_n$.
2. $\mathcal{L}_{n+1}$ includes all the lines of $\mathcal{L}_n$, together with a new and distinct line joining each pair of points in $\mathcal{P}_n$ which do not already have a line joining them in $\mathcal{C}_n$.
3. $\mathbf{I}_{n+1}$ is $\mathbf{I}_n$, plus the incidences involving the added points and lines described above.

Let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$, $\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n$, and $\mathbf{I} = \bigcup_{n=0}^{\infty} \mathbf{I}_n$. Then the incidence structure $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ is the free completion of the configuration $\mathcal{C}_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathbf{I}_0)$.

We showed in class that if the initial configuration was a set of four points such that no three were on the same line, then the free completion of the configuration is a projective plane.

1. Suppose $\mathcal{C}_0 = (\{A, B, C, D\}, \emptyset, \emptyset)$ and $\sigma : \{A, B, C, D\} \to \{A, B, C, D\}$ is any permutation of $\{A, B, C, D\}$. Show that σ can be extended to a collineation of the free completion of $\mathcal{C}_0$. *[10]*

Note. It's harder to prove, but all the collineations of a free completion come from incidence-preserving permutations of the initial configuration.

Recall that we're switching the due dates for our weekly assignments to Mondays instead of Thursdays so that we can make more effective use of our seminars.