Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry
TRENT UNIVERSITY, Fall 2019

Assignment #4
Collineations
Due on Thursday, 3 October.

Recall from class that a *collineation* of a projective plane is a 1-1 onto function α that takes points of the projective plane to points of the projective plane, lines of the projective plane to lines of the projective plane, and preserves incidence. *i.e.* $P \, \mathbf{I} \, \ell \Leftrightarrow P^\alpha \, \mathbf{I} \, \ell^\alpha$. It is traditional to write P^α for $\alpha(P)$ and similarly for lines; compositions of collineations work "inside first" in this notation, *e.g.*. $P^{\alpha\beta} = P^{\beta\circ\alpha} = \beta\left(\alpha(P)\right)$.

1. Show that if α is a collineation of a projective plane, then α^{-1} is also a collineation of the same projective plane. *[2]*

2. Show that if α and β are collineations of a projective plane, then $\alpha\beta = \beta \circ \alpha$ is also a collenation of the same projective plane. *[2]*

NOTE. For those of you taking abstract algebra, the two problems above do most of the work in showing that the collineations of a projective plane form a group, with the group operation being composition.

Recall also that a collineation is said to be (P, ℓ)-*central* (sometimes referred to as a (P, ℓ)-*perspectivity*) if it has has centre P and axis ℓ, *i.e.* $Q^\alpha = Q$ for every point Q on the axis ℓ and $m^\alpha = m$ for every line m passing through the centre P. (We showed in class that a collineation has an axis if and only if it has a centre.) A (P, ℓ)-central collineation is said to be an *elation* if P is on ℓ, and is said to be a *homology* if P is not on ℓ.

DEFINITION. Two triangles ABC and DEF are said to be *in perspective from a point* P if AD, BE, and CF are all incident with P, and *in perspective from a line* ℓ if $AB \cap DE$, $BC \cap EF$, and $AE \cap DF$ are all incident with ℓ.

3. Suppose α is a (P, ℓ)-central collineation of a projective plane and ABC is a triangle of the projective plane such that none of A, B, and C are P or incident with ℓ. Show that ABC and $A^\alpha B^\alpha C^\alpha$ are in perspective from P and in perspective from ℓ. *[3]*

4. Suppose α is a (P, ℓ)-central collineation of a projective plane and A and B are points of the projective plane which are not on ℓ and such that A, B, and P are not collinear, and that we know A^α and B^α. Show that this completely determines *alpha*, *i.e.* given any point C in the plane, show how to find C^α, and given any line m, show how to find m^α. *[3]*