Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry
TRENT UNIVERSITY, Fall 2019

Assignment #3
Collineations of the Real Projective Plane from Linear Algebra
Due on Thursday, 26 September.

Recall from class that we can, among other ways, define the real projective plane using *projective coordinates*:

- Points are represented by non-zero vectors $(a, b, c) \in \mathbb{R}^3$, and another vector (d, e, f) represents the same point if there is a scalar $\lambda \neq 0$ such that $(a, b, c) = \lambda(d, e, f)$.
- Lines are represented by non-zero vectors $[p, q, r] \in \mathbb{R}^3$, and another vector $[s, t, u]$ represents the same point if there is a scalar $\lambda \neq 0$ such that $[p, q, r] = \lambda[s, t, u]$.
- A point (a, b, c) is incident with a line $[p.q.r]$, often written as $(a, b, c)\,\mathbf{I}\,[p.q.r]$, if and only if $(a, b, c) \cdot [p, q, r] = ap + bq + cr = 0$.

Suppose $\mathbf{A}$ is a 3×3 invertible matrix with real entries. Define a function φ that maps points of the real projective plane to points of the real projective plane by $\varphi(P) = \left(\mathbf{M}P^T\right)^T = P\mathbf{M}^T$. (The transposes are there because points are represented by row vectors and matrix multiplication is commonly defined for column vectors.)

1. Verify that φ does indeed take points of the real projective plane to points of the real projective plane, and is also 1–1 and onto. *[5]*

2. Find a way to define φ on the lines so that is a $1 - 1$ onto function that takes lines of the real projective plane to lines of the real projective plane and also preserves incidence, *i.e.* has $\varphi(P)\,\mathbf{I}\,\varphi(\ell) \Leftrightarrow P\,\mathbf{I}\,\ell$ for all point P and lines ℓ of the real projective plane. Verify that your definition does the job! *[5]*