Mathematics 3260H – Geometry II: Projective and non-Euclidean geometry
TRENT UNIVERSITY, Fall 2017

Take-home Final Exam

Due on ~~Friday~~ Saturday, 16 December, 2017.

Instructions: Do all three of parts /, //, and ///, and, if you wish, part \ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, you may not consult any other sources, nor consult or work with any other person on this exam.

Part /. Do *all* four (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Suppose that $\triangle ABC$ in the elliptic plane has $\angle ACB = \frac{\pi}{2}$. Show that
$$\cos(\angle ABC) = \sin(\angle CAB)\cos(|AC|).$$

2. Suppose that $\triangle ABC$ in the hyperbolic plane has $\angle ACB = \frac{\pi}{2}$. Show that
$$\cos(\angle ABC) = \sin(\angle CAB)\cosh(|AC|).$$

3. Suppose γ is a collineation of a projective plane which is its own inverse, *i.e.* $\gamma^2 = \gamma \circ \gamma$ is the identity. Show that γ must fix more than one point and more than one line.

4. The antipodal sphere model of the elliptic plane is an instance of the real projective plane. If one removes a line and all of the points on it from the real projective plane, one is left with the real affine plane, *i.e.* the Euclidean plane. Thus the Euclidean plane is embedded in a model of the elliptic plane, even though these have very different properties in some respects. Explain why this isn't a problem.

Part //. Do any *two* (2) of problems **5** – **8**. *[20 = 2 × 10 each]*

5. Using Euclid's Postulates I–IV, show that Playfair's Postulate (*i.e.* **AII**) is true if and only if the sum of the interior angles of any quadrilateral is is 2π. (You may use those results in Book I of the *Elements* that depend only on Postulates I–IV.)

6. Determine whether the Side-Angle-Side-Angle-Side congruence criterion for quadrilaterals is true in each of the elliptic, Euclidean, and hyperbolic planes.

7. Is it possible to tile the hyperbolic plane with a quadrilateral? (That is, cover all of the hyperbolic plane with congruent copies of the same quadrilateral, with no overlap between the copies except for the sides and vertices.) Give an example of such a tiling or explain why there can't be one.

8. Is Desargues' Theorem true in the hyperbolic plane? Show that it is or give a counterexample demonstrating that it can fail.

Parts /// and \ are on page 2.

Part ///. Do any *two* (2) of problems **9** – **12**. *[20 = 2 × 10 each]*

9. Show that Desargues' Theorem is not always true in the Moulton plane.

10. Let $\mathbb{Z}_5$ be the field with five elements, *i.e.* $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, with $+$ and $\cdot$ done modulo 5. Determine whether the projective plane with $\mathbb{Z}_5$ as its underlying algebraic structure has a (P, ℓ)-collineation for some P not incident with ℓ or not.

11. Suppose δ is a (P, ℓ)-collineation and γ is a (Q, ℓ)-collineation, where P and Q are points and ℓ is a line of the projective plane in question. Show that $\gamma\delta = \delta \circ \gamma$ is an (R, ℓ)-collineation for some point R and, if you can, show how to find the centre R.

12. Suppose one replaces axiom **AII** of an affine plane (*i.e.* Playfair's Axiom) with its hyperbolic counterpart:

> **HII.** *Given any line ℓ and any point P not on ℓ, there is more than one line through P which is parallel to ℓ.*

Give an example of a *finite* incidence structure satisfying **AI**, **HII**, and **AIII**, or show that there can be no such finite structure.

Part \\.

- Write a poem about geometry or mathematics in general. *[1]*

[Total = 80]

I HOPE THAT YOU ENJOYED THIS COURSE.
HAVE A GOOD BREAK!