Hilbert's Axioms for Euclidean Geometry

Let us consider three distinct systems of things. The things composing the first system, we will call *points* and designate them by the letters A, B, C, ... ; those of the second, we will call *straight lines* and designate them by the letters a, b, c, ... ; and those of the third system, we will call *planes* and designate them by the Greek letters α, β, γ, ... The points are called the *elements of linear geometry*; the points and straight lines, the *elements of plane geometry*; and the points, lines, and planes, the *elements of the geometry of space* or the *elements of space*.

We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated," "between," "parallel," "congruent," "continuous," etc. The complete and exact description of these relations follows as a consequence of the *axioms of geometry*. These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition.*

I. Axioms of Incidence

I.1 Given any two points, there exists a line containing both of them.

I.2 Given any two points, there exists no more than one line containing both points.

I.3 A line contains at least two points, and given any line, there exists at least one point not on it.

I.4 Given any three points not contained in one line, there exists a plane containing all three points. Every plane contains at least one point.

I.5 Given any three points not contained in one line, there exists only one plane containing all three points.

I.6 If two points contained in line m lie in some plane α, then α contains every point in m.

I.7 If the planes α and β both contain the point A, then α and β both contain at least one other point.

I.8 There exist at least four points not all contained in the same plane.

II. Axioms of Order

II.1 If a point B is between points A and C, B is also between C and A, and there exists a line containing the points A, B, C.

II.2 Given two points A and C, there exists a point B on the line AC such that C lies between A and B.

II.3 Given any three points contained in one line, one and only one of the three points is between the other two.

II.4 Given any four points on a line, it is always possible to assign them the names A, B, C, and D, such that B is between A and C and A and D. Likewise, C will be between A and D and also between B and D.

* From Chapter I of *Foundations of Geometry* by David Hilbert.

II.5 *(Pasch's Axiom.)* Given three points A, B, C not contained in one line, and given a line m contained in the plane ABC but not containing any of A, B, C: if m contains a point on the segment AB, then m also contains a point on the segment AC or on the segment BC.

III. Axiom of Parallels

III.1 *(Playfair's Postulate.)* Given a line m, a point A not on m, and a plane containing both m and A: in that plane, there is at most one line containing A and not containing any point on m.

IV. Axioms of Congruence

IV.1 Given two points A, B, and a point A' on line m, there exist two and only two points C and D, such that A' is between C and D, and $AB \cong A'C$ and $AB \cong A'D$.

IV.2 If $CD \cong AB$ and $EF \cong AB$, then $CD \cong EF$.

IV.3 Let line m include the segments AB and BC whose only common point is B, and let line m or m' include the segments $A'B'$ and $B'C'$ whose only common point is B'. If $AB \cong A'B'$ and $BC \cong B'C'$ then $AC \cong A'C'$.

IV.4 Given the angle $\angle ABC$ and ray $B'C'$, there exist two and only two rays, $B'D$ and $B'E$, such that $\angle DB'C' \cong \angle ABC$ and $\angle EB'C' \cong \angle ABC$.

IV.5 Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $AB \cong A'B'$, $AC \cong A'C'$, and $\angle BAC \cong \angle B'A'C'$, then $\triangle ABC \cong \triangle A'B'C'$.

V. Axioms of Continuity

V.1 *(Axiom of Archimedes.)* Given the line segment CD and the ray AB, there exist n points $A_1, \ldots, A_n$ on AB, such that $A_j A_{j+1} \cong CD$, $1 \le j \le n$. Moreover, B is between A_1 and A_n.

V.2 *(Line completeness.)* Adding points to a line results in an object that violates one or more of the following axioms: **I**, **II**, **III.1-2**, **V.1**.

Adapted from the article *Hilbert's Axioms* on Wikipedia, which can be found at

`http://en.wikipedia.org/wiki/Hilbert's_axioms` ,

and David Hilbert's book *Foundations of Geometry*, which can be downloaded in pdf format from Project Gutenberg at

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