Mathematics 3200H – Number Theory
Trent University, Fall 2022

Take-Home Final Examination
Due on Friday, 16 December.

Instructions: Do both of parts **P** and **Q**, and, if you wish, part **R** as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. In particular, you may use all the resources this course has on Blackboard and on its archive page. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part Prime. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. When the students in the class lined up four to a row, there was one left over; when they lined up five to a row, there were two left over; and when they lined up seven to a row, there were three left over. How many students are in the class?

2. Suppose p is an odd prime and g and h are primitive roots of p. Show that for some odd integer k, $h \equiv g^k \pmod{p}$.

3. Show that $\phi(n) \neq 14$ for all $n \geq 1$.

4. Suppose p and q are odd primes such that $p = q + 8a$ for some positive integer a. Show that $\left(\frac{p}{q}\right) = \left(\frac{a}{q}\right) = \left(\frac{a}{p}\right)$.

Part Quotient. Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Find a positive integer n such that $\frac{n}{2}$ is a square, $\frac{n}{3}$ is a cube, and $\frac{n}{5}$ is a fifth power.

6. Suppose that $n \equiv 4 \pmod{9}$. Show that n cannot be written as the sum of three cubes.

7. Suppose p is an odd prime. Show that $(1^p + 2^p + 3^p + \cdots + (p-1)^p) \equiv 0 \pmod{p}$.

8. Suppose m and n are positive integers. Is it true that if m and mn are each a sum of two integer squares, then so is n? Prove it or give a counterexample.

9. Show that if $2^n - 1$ is a prime number, then n is a prime number.

10. Show that any odd perfect number would have to be of the form $p^{4k+1}m^2$, where p is an odd prime and k and m are integers.

11. Recall that an integer m is a square if $m = k^2$ for some integer k and is a triangular number if $m = n(n+1)/2$ for some integer n. Determine how many integers are simultaneously square and triangular.

[Total = 80]

Part Rhyme. Bonuses!

12. Write an original poem about number theory. *[1]*

13. Suppose $n = x^2 + y^2 + z^2 + w^2$ for some x, y, z, $w \in \mathbb{Z}$. Explain why we can assume that $3 \mid (x + y + z)$. *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE. HAVE A GOOD BREAK!