Mathematics 3200H – Number Theory
TRENT UNIVERSITY, FALL 2015
Take-Home Final Examination
Due on Friday, 18 December, 2015.

Instructions: Do both of parts **P** and **Q**, and, if you wish, part **R** as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part Prime. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Is $f(n) = n^2 + n + 41$ prime for all $n \in \mathbb{N}$? Prove it is or give a counterexample.

2. Suppose p and q are odd primes such that $p = q + 4a$ for some positive integer a. Show that $\left(\frac{p}{q}\right) = \left(\frac{a}{q}\right) = \left(\frac{a}{p}\right)$.

3. Show that if $n \equiv 4 \pmod 9$, then n can't be written as the sum of three cubes.

4. Suppose $n > 0$ and let $p = 3 \cdot 2^n - 1$, $q = 3 \cdot 2^{n-1} - 1$, and $r = 9 \cdot 2^{2n-1} - 1$. Show that if p, q, and r are all prime, then $2^n pq$ and $2^n r$ are an amicable pair, *i.e.* $\sigma(2^n pq) = 2^n pq + 2^n r = \sigma(2^n r)$.

Part Qrime? Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Find a positive integer n such that $\frac{n}{2}$ is a square, $\frac{n}{3}$ is a cube, and $\frac{n}{5}$ is a fifth power.

6. Show that $\phi(n) \neq 14$ for all $n \geq 1$.

7. Let p be an odd prime. For which n does $\left(1 + n + n^2 + \cdots + n^{p-2}\right) \equiv 0 \pmod p$? Prove it.

8. Suppose m and n are positive integers. Is it true that if any two of m, n, and mn are sums of two integer squares, then so is the third? Prove it or give a counterexample.

9. Suppose g and h are primitive roots of an odd prime p. Show that $h \equiv g^{2k+1} \pmod p$ for some integer k.

10. Show that any odd perfect number would have to be of the form $p^{4k+1}m^2$, where p is an odd prime and k and m are integers.

11. Determine what condition(s) r must satisfy to ensure that if a is a quadratic residue mod p and $ab \equiv r \pmod p$, it follows that b is also a quadratic residue mod p.

[Total = 80]

Part Rhyme! Bonus . . .

0. Write an original poem about number theory. *[1]*

-1. Suppose $n = x^2 + y^2 + z^2 + w^2$ for some x, y, z, $w \in \mathbb{Z}$. Explain why we can assume that $3 \mid (x + y + z)$. *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE. HAVE A GOOD BREAK!