Mathematics 235H – Linear algebra II: Vector spaces
TRENT UNIVERSITY, Winter 2008

Solutions to Assignment #9

Vector spaces of linear transformations

Recall from class that if U and V are vector spaces (using the same set of scalars), then

$$L(U,V) = \{\, T \mid T \text{ is a linear transformation } U \to V \,\}$$

is itself a vector space (using the same scalars as U and V) when vector addition is defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

and scalar multiplication is defined by

$$(cT)(\mathbf{u}) = cT(\mathbf{u})\,.$$

1. Suppose that $\dim(U) = 3$ and $\dim(V) = 4$. What is $\dim\left(L(U,V)\right)$? Explain why! *[5]*

Solution. If $\dim(U) = 3$ and $\dim(V) = 4$, then $\dim\left(L(U,V)\right) = 12$. To see this, it suffices to produce a basis for $L(U,V)$ and observe that this basis has twelve elements,

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for U – recall that $\dim(U) = 3$ – and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$ be a basis for V – recall that $\dim(V) = 4$. For each $1 \le i \le 3$ and $1 \le j \le 4$, define the linear transformation $T_{ij} : U \to V$ as follows:

$$T_{ij}\left(\mathbf{b}_k\right) = \begin{cases} \mathbf{c}_j & \text{if } i = k \\ \mathbf{0} & \text{if } i \neq k \end{cases}$$

That is, T_{ij} sends $\mathbf{b}_i$ to $\mathbf{c}_j$ and sends the other basis vectors to $\mathbf{0}$. (Recall that to define a linear transformation it is sufficient to specify what it does on each basis element of the domain vector space, which in this case is U.) We will show that $\mathcal{T} = \{\, T_{ij} \mid 1 \le i \le 3 \text{ and } 1 \le j \le 4 \,\}$ is a basis for $L(U,V)$.

First, to see that $\mathcal{T}$ spans $L(U,V)$, suppose $T : U \to V$ is a linear transformation and suppose that

$$\begin{aligned} T(\mathbf{b}_1) &= a_{11}\mathbf{c}_1 + a_{12}\mathbf{c}_2 + a_{13}\mathbf{c}_3 + a_{14}\mathbf{c}_4\,, \\ T(\mathbf{b}_2) &= a_{21}\mathbf{c}_1 + a_{22}\mathbf{c}_2 + a_{23}\mathbf{c}_3 + a_{24}\mathbf{c}_4\,, \text{ and} \\ T(\mathbf{b}_3) &= a_{31}\mathbf{c}_1 + a_{32}\mathbf{c}_2 + a_{33}\mathbf{c}_3 + a_{34}\mathbf{c}_4\,, \end{aligned}$$

where the a_{ij} are scalars. We claim that $T = a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34}$; to verify this it is sufficient to check that T is equal to the sum on each of the basis vectors

$\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. One:

$$\begin{aligned}
&(a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34})\,(\mathbf{b}_1) \\
=\,& a_{11}T_{11}(\mathbf{b}_1) + a_{12}T_{12}(\mathbf{b}_1) + a_{13}T_{13}(\mathbf{b}_1) + a_{14}T_{14}(\mathbf{b}_1) \\
&+ a_{21}T_{21}(\mathbf{b}_1) + a_{22}T_{22}(\mathbf{b}_1) + a_{23}T_{23}(\mathbf{b}_1) + a_{24}T_{24}(\mathbf{b}_1) \\
&+ a_{11}T_{11}(\mathbf{b}_1) + a_{12}T_{12}(\mathbf{b}_1) + a_{13}T_{13}(\mathbf{b}_1) + a_{14}T_{34}(\mathbf{b}_1) \\
=\,& a_{11}\mathbf{c}_1 + a_{12}\mathbf{c}_2 + a_{13}\mathbf{c}_3 + a_{14}\mathbf{c}_4 + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} \\
=\,& T(\mathbf{b}_1)
\end{aligned}$$

Two:

$$\begin{aligned}
&(a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34})\,(\mathbf{b}_2) \\
=\,& a_{11}T_{11}(\mathbf{b}_2) + a_{12}T_{12}(\mathbf{b}_2) + a_{13}T_{13}(\mathbf{b}_2) + a_{14}T_{14}(\mathbf{b}_2) \\
&+ a_{21}T_{21}(\mathbf{b}_2) + a_{22}T_{22}(\mathbf{b}_2) + a_{23}T_{23}(\mathbf{b}_2) + a_{24}T_{24}(\mathbf{b}_2) \\
&+ a_{11}T_{11}(\mathbf{b}_2) + a_{12}T_{12}(\mathbf{b}_2) + a_{13}T_{13}(\mathbf{b}_2) + a_{14}T_{34}(\mathbf{b}_2) \\
=\,& \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + a_{21}\mathbf{c}_1 + a_{22}\mathbf{c}_2 + a_{23}\mathbf{c}_3 + a_{24}\mathbf{c}_4 + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} \\
=\,& T(\mathbf{b}_2)
\end{aligned}$$

Three:

$$\begin{aligned}
&(a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34})\,(\mathbf{b}_3) \\
=\,& a_{11}T_{11}(\mathbf{b}_3) + a_{12}T_{12}(\mathbf{b}_3) + a_{13}T_{13}(\mathbf{b}_3) + a_{14}T_{14}(\mathbf{b}_3) \\
&+ a_{21}T_{21}(\mathbf{b}_3) + a_{22}T_{22}(\mathbf{b}_3) + a_{23}T_{23}(\mathbf{b}_3) + a_{24}T_{24}(\mathbf{b}_3) \\
&+ a_{11}T_{11}(\mathbf{b}_3) + a_{12}T_{12}(\mathbf{b}_3) + a_{13}T_{13}(\mathbf{b}_3) + a_{14}T_{34}(\mathbf{b}_3) \\
=\,& \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + a_{31}\mathbf{c}_1 + a_{32}\mathbf{c}_2 + a_{33}\mathbf{c}_3 + a_{34}\mathbf{c}_4 \\
=\,& T(\mathbf{b}_3)
\end{aligned}$$

Thus $\mathcal{T}$ spans $L(U, V)$.

Second, recall that the zero vector of $L(u, V)$ is the linear transformation O such that $O(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in U$. To check that $\mathcal{T}$ is linearly independent, we need to check that if $a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34} = O$ for scalars a_{11}, a_{12}, ..., a_{33}, a_{34}, then $a_{11} = a_{12} = \cdots = a_{33} = a_{34} = 0$. If $a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34} = O$, then, in particular,

$$\begin{aligned}
&(a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34})\,(\mathbf{b}_1) \\
=\,& a_{11}T_{11}(\mathbf{b}_1) + a_{12}T_{12}(\mathbf{b}_1) + a_{13}T_{13}(\mathbf{b}_1) + a_{14}T_{14}(\mathbf{b}_1) \\
&+ a_{21}T_{21}(\mathbf{b}_1) + a_{22}T_{22}(\mathbf{b}_1) + a_{23}T_{23}(\mathbf{b}_1) + a_{24}T_{24}(\mathbf{b}_1) \\
&+ a_{11}T_{11}(\mathbf{b}_1) + a_{12}T_{12}(\mathbf{b}_1) + a_{13}T_{13}(\mathbf{b}_1) + a_{14}T_{34}(\mathbf{b}_1) \\
=\,& a_{11}\mathbf{c}_1 + a_{12}\mathbf{c}_2 + a_{13}\mathbf{c}_3 + a_{14}\mathbf{c}_4 + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} \\
=\,& O(\mathbf{b}_1) = \mathbf{0}\,.
\end{aligned}$$

Since $\mathcal{C}$ is a basis for V, it follows from $a_{11}\mathbf{c}_1 + a_{12}\mathbf{c}_2 + a_{13}\mathbf{c}_3 + a_{14}\mathbf{c}_4 = \mathbf{0}$ that $a_{11} = a_{12} =$

$a_{13} = a_{14} = 0$. Similarly,

$$\begin{aligned}
&(a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34})(\mathbf{b}_2) \\
&= a_{11}T_{11}(\mathbf{b}_2) + a_{12}T_{12}(\mathbf{b}_2) + a_{13}T_{13}(\mathbf{b}_2) + a_{14}T_{14}(\mathbf{b}_2) \\
&\quad + a_{21}T_{21}(\mathbf{b}_2) + a_{22}T_{22}(\mathbf{b}_2) + a_{23}T_{23}(\mathbf{b}_2) + a_{24}T_{24}(\mathbf{b}_2) \\
&\quad + a_{11}T_{11}(\mathbf{b}_2) + a_{12}T_{12}(\mathbf{b}_2) + a_{13}T_{13}(\mathbf{b}_2) + a_{14}T_{34}(\mathbf{b}_2) \\
&= \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + a_{21}\mathbf{c}_1 + a_{22}\mathbf{c}_2 + a_{23}\mathbf{c}_3 + a_{24}\mathbf{c}_4 + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} \\
&= O(\mathbf{b}_2) = \mathbf{0}\,.
\end{aligned}$$

Since $\mathcal{C}$ is a basis for V, it follows from $a_{21}\mathbf{c}_1 + a_{22}\mathbf{c}_2 + a_{23}\mathbf{c}_3 + a_{24}\mathbf{c}_4 = \mathbf{0}$ that also $a_{21} = a_{22} = a_{23} = a_{24} = 0$. Again, similarly,

$$\begin{aligned}
&(a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{33}T_{33} + a_{34}T_{34})(\mathbf{b}_3) \\
&= a_{11}T_{11}(\mathbf{b}_3) + a_{12}T_{12}(\mathbf{b}_3) + a_{13}T_{13}(\mathbf{b}_3) + a_{14}T_{14}(\mathbf{b}_3) \\
&\quad + a_{21}T_{21}(\mathbf{b}_3) + a_{22}T_{22}(\mathbf{b}_3) + a_{23}T_{23}(\mathbf{b}_3) + a_{24}T_{24}(\mathbf{b}_3) \\
&\quad + a_{11}T_{11}(\mathbf{b}_3) + a_{12}T_{12}(\mathbf{b}_3) + a_{13}T_{13}(\mathbf{b}_3) + a_{14}T_{34}(\mathbf{b}_3) \\
&= \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + a_{31}\mathbf{c}_1 + a_{32}\mathbf{c}_2 + a_{33}\mathbf{c}_3 + a_{34}\mathbf{c}_4 \\
&= O(\mathbf{b}_3) = \mathbf{0}
\end{aligned}$$

Since $\mathcal{C}$ is a basis for V, it follows from $a_{31}\mathbf{c}_1 + a_{32}\mathbf{c}_2 + a_{33}\mathbf{c}_3 + a_{34}\mathbf{c}_4 = \mathbf{0}$ that also $a_{31} = a_{32} = a_{33} = a_{34} = 0$. Hence $\mathcal{T}$ is linearly independent.

Thus $\mathcal{T}$ is a basis for $L(U, V)$, and, since it has twelve elements, it it folows that $\dim(L(U, V)) = 12$. ■

2. Determine whether each of the following sets is a subspace of $L(U, V)$ or not.

a. $W = \{\, T \in L(U, V) \mid T \text{ is one-to-one} \,\}$ *[1]*

b. $W = \{\, T \in L(U, V) \mid T \text{ is onto} \,\}$ *[1]*

c. $W = \{\, T \in L(U, V) \mid T \text{ is invertible} \,\}$ *[1]*

d. $W = \{\, T \in L(U, V) \mid T \text{ is not invertible} \,\}$ *[2]*

Solution. Recall that the zero vector of $L(U, V)$ is the linear transformation O such that $O(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in U$. The zero vector must be in every subspace, which eliminates the would-be subspaces in **a**–**c** most of the time:

First, $\ker(O) = U$, so O is one-to-one only if $U = \{\mathbf{0}\}$.

Second, $\operatorname{ran}(O) = \{\mathbf{0}\}$, so O is onto only if $V = \{\mathbf{0}\}$.

Third, O is invertible exactly when it is both one-to-one and onto, *i.e.* if both $U = \{\mathbf{0}\}$ and $V = \{\mathbf{0}\}$.

Note that in the exceptional situations where $U = \{\mathbf{0}\}$, $V = \{\mathbf{0}\}$, or both, O is, respectively, one-to-one, onto, or invertible. In these cases, W is a subspace of $L(U, V)$, since it is, respectively, $\{O\}$, $L(U, V)$, or $\{O\} = L(U, V)$. (Check this for yourselves!)

For **d**, observe that unless $\{O\} = L(U, V)$ (in which case W is not a subspace because it is empty), O is not invertible and hence is in W. However, W may or may not be subspace even so:

First, for there to be any invertible linear transformations between U and V, we must have $\dim(U) = \dim(V)$, so if $\dim(U) \neq \dim(V)$, then $W = L(U, V)$ and so is a subspace of $L(U, V)$.

Second, if $\dim(U) = \dim(V) = 1$, the only non-invertible linear transformation is O, so $W = \{O\}$ is a subspace of $L(U, V)$.

Third, if $\dim(U) = \dim(V) > 1$, W will not be a subspace because it will be possible to have two non-invertible linear transformations add up to an invertible linear transformation. If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \dots\}$ are bases of U and V respectively, then neither of the linear transformations defined by

$$S(\mathbf{b}_i) = \begin{cases} \mathbf{c}_1 & \text{if } i = 1 \\ \mathbf{0} & \text{if } i > 1 \end{cases} \qquad \text{and} \qquad T(\mathbf{b}_i) = \begin{cases} \mathbf{0} & \text{if } i = 1 \\ \mathbf{c}_i & \text{if } i > 1 \end{cases}$$

is invertible – and hence they're in W – but their sum, which is given by

$$(S + T)(\mathbf{b}_i) = \mathbf{c}_i \qquad \text{for each } i \geq 1,$$

is indeed invertible (why?) – and hence is not in W. Since W is not closed under addition, it can't be a subspace. ■