

Mathematics 235H – Linear algebra II: Vector spaces

TRENT UNIVERSITY, Winter 2008

Solutions to Assignment #11

Moving triangles in perspective

Two triangles $\triangle ABC$ and $\triangle XYZ$ are in *perspective* from a point P if the lines AX , BY , and CZ all pass through P . We'll be looking at moving triangles in perspective from the origin in \mathbb{R}^3 using linear transformations. In what follows let $\triangle ABC$ have vertices $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$, and let $\triangle XYZ$ have vertices $X = (2, 0, 0)$, $Y = (0, 3, 0)$, and $Z = (0, 0, 4)$. It's obvious that $\triangle ABC$ and $\triangle XYZ$ are in perspective from the origin.

1. Find an invertible linear transformation from \mathbb{R}^3 to \mathbb{R}^3 that moves $\triangle ABC$ to $\triangle XYZ$. [3]

Solution. The linear transformation with matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ (relative to the standard basis for \mathbb{R}^3) does the job for the vertices of the triangle:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

For those who care, it's not hard to check that if a linear transformation moves the vertices of one triangle to those of another triangle, then it also moves the corresponding sides accordingly.

This linear transformation is invertible because its matrix is invertible, which is the case since

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2 \cdot 3 \cdot 4 = 24 \neq 0. \quad \blacksquare$$

2. In general, if $\triangle DEF$ and $\triangle RST$ are in perspective from the origin and are not in the same plane, show there must be an invertible linear transformation from \mathbb{R}^3 to \mathbb{R}^3 that moves $\triangle DEF$ to $\triangle RST$. [3]

Solution. Suppose $\triangle DEF$ and $\triangle RST$ are in perspective from the origin and are not in the same plane. It follows that $\triangle DEF$ cannot be in the same plane with the origin, because if it were, the requirement that $\triangle DEF$ and $\triangle RST$ be in perspective from the origin would require both of them to be in this plane. $\triangle RST$ cannot be in the same plane with the origin for the same reason. Note that it follows from these facts that none of the vertices of either triangle can be the origin.

Since $\triangle DEF$ is not in the same plane with the origin O , the vectors \overrightarrow{OD} , \overrightarrow{OE} , and \overrightarrow{OF} must be linearly independent and hence form a basis for \mathbb{R}^3 . Define a linear transformation

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(\overrightarrow{OD}) = \overrightarrow{OR}$, $T(\overrightarrow{OE}) = \overrightarrow{OS}$, and $T(\overrightarrow{OF}) = \overrightarrow{OT}$. (Recall that to define a linear transformation it is sufficient to define it on a basis.) This transformation clearly takes the vertices of $\triangle DEF$ to the corresponding vertices of $\triangle RST$.

To see that T is also invertible, note that since $\triangle RST$ is not in the same plane with the origin O , the vectors \overrightarrow{OR} , \overrightarrow{OS} , and \overrightarrow{OT} must be also linearly independent and hence form a basis for \mathbb{R}^3 . Thus T takes a basis into a basis, so it must be invertible. (Why does this follow?) ■

3. Find the coordinates of the points where the line AB meets the line XY , the line AC meets the line XZ , and the line BC meets the line YZ , respectively. Is there anything special about these points? [4]

Solution. $A = (1, 0, 0)$, $B = (0, 1, 0)$, $X = (2, 0, 0)$, and $Y = (0, 3, 0)$ are all in the plane $z = 0$, *i.e.* the xy -plane. We can therefore forget about the third coordinate – the intersection of AB with XY will have to have third coordinate 0 as well – and suppose $A = (1, 0)$, $B = (0, 1)$, $X = (2, 0)$, and $Y = (0, 3)$. The line joining $(1, 0)$ to $(0, 1)$ in the xy -plane is the line with equation $y = -x + 1$, and the line joining $(2, 0)$ to $(0, 3)$ is the line with equation $y = -\frac{3}{2}x + 3$. These lines meet for the x such that $-x + 1 = -\frac{3}{2}x + 3$, rearranging this gives $\frac{1}{2}x = -x + 2 = 3 - 1 = 2$, and so $x = 2 \cdot 2 = 4$. The corresponding value of y is $y = -4 + 1 = -3$. Thus the intersection of XY with AB has coordinates $(4, -3, 0)$.

Similarly, $A = (1, 0, 0)$, $C = (0, 0, 1)$, $X = (2, 0, 0)$, and $Z = (0, 0, 4)$ are all in the plane $y = 0$, *i.e.* the xz -plane. We can therefore forget about the second coordinate – the intersection of AC with XZ will have to have second coordinate 0 as well – and suppose $A = (1, 0)$, $C = (0, 1)$, $X = (2, 0)$, and $Z = (0, 4)$. The line joining $(1, 0)$ to $(0, 1)$ in the xz -plane is the line with equation $z = -x + 1$, and the line joining $(2, 0)$ to $(0, 4)$ is the line with equation $z = -2x + 4$. These lines meet for the x such that $-x + 1 = -2x + 4$, rearranging this gives $x = -x + 3 = 4 - 1 = 3$, and so $x = 3$. The corresponding value of z is $z = -3 + 1 = -2$. Thus the intersection of XZ with AC has coordinates $(3, 0, -2)$.

Similarly (again!), $B = (0, 1, 0)$, $C = (0, 0, 1)$, $Y = (0, 3, 0)$, and $Z = (0, 0, 4)$ are all in the plane $x = 0$, *i.e.* the yz -plane. We can therefore forget about the first coordinate – the intersection of BC with YZ will have to have first coordinate 0 as well – and suppose $B = (1, 0)$, $C = (0, 1)$, $Y = (3, 0)$, and $Z = (0, 4)$. The line joining $(1, 0)$ to $(0, 1)$ in the yz -plane is the line with equation $z = -y + 1$, and the line joining $(2, 0)$ to $(0, 4)$ is the line with equation $z = -\frac{4}{3}y + 4$. These lines meet for the y such that $-y + 1 = -\frac{4}{3}y + 4$, rearranging this gives $\frac{1}{3}y = -y + 3 = 4 - 1 = 3$, and so $y = 3 \cdot 3 = 9$. The corresponding value of z is $z = -9 + 1 = -8$. Thus the intersection of YZ with BC has coordinates $(0, 9, -8)$.

What is special about these points? They are *collinear*, *i.e.* on the same straight line. This can be seen by considering the vectors between them. In particular, the vector with its tail at $(4, -3, 0)$ and tip at $(3, 0, -2)$ is $\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$, and the vector with its tail at $(3, 0, -2)$ and its tip at $(0, 9, -8)$ is $\begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}$. Since the latter vector is a multiple of the former, the three points are on the same straight line. ■