Mathematics 235H – Linear algebra II: Vector spaces
TRENT UNIVERSITY, Winter 2008

Solutions to Assignment #1

The Island of Knights and Knaves

The inhabitants of the Island of Knights and Knaves are either knights, who always tell the truth, or knaves, who always lie. While visiting the Island you encounter . . .

1. . . . a fork in the road with a tourist information booth staffed by two inhabitants. A sign on the booth tells you that:

i. One of the forks leads to the the Hamlet of Indeterminacy, but doesn't tell you which one.
ii. One of the staffers is a knight and the other is a knave, but you only get to ask one of them a single question.

What should you ask one of the staffers to find out which fork of the road leads to the Hamlet of Indeterminacy? *[3]*

Solution. You should ask something like "If I asked your fellow staffer which fork in the road leads to the Hamlet of Indeterminacy, what answer would I receive?" of either of the staffers, and then take the *other* fork. Here's why:

If the staffer you ask is a knight, he will tell the truth about what the other staffer would answer if asked which fork in the road leads to the Hamlet of Indeterminacy. The other staffer, however, being a knave, would lie about which fork in the road leads to the Hamlet of Indeterminacy, so the (knightly) staffer you asked your question of would pass on this lie.

On the other hand, if the staffer you ask is a knave, he will lie about what the other staffer would answer if asked which fork in the road leads to the Hamlet of Indeterminacy. The other staffer, however, being a knight, would tell the truth about which fork in the road leads to the Hamlet of Indeterminacy, but then the (knavely) staffer you asked your question of would lie about what the other would say.

Either way, the indicated fork in the road is the wrong one, so you need to take the other one. ■

2. . . . two inhabitants, A and B, who tell you the following.

A: Both of us are knights.
B: One of us is a knight and the other is a knave.

Determine, if you can, whether each of A and B is a knight or a knave. *[3]*

Solution. A must be a knave, but B could be either a knight or a knave – we don't have enough information to determine which. Here's the reasoning:

Suppose A were a knight. Then, since knights only speak the truth, B must be a knight as well. However, this contradicts the fact that B's statement is incorrect if both A and B are knights. Since the assumption that A is a knight leads to a contradiction, A must be a knave.

To see that B could be either a knight or a knave, observe that if B were a knight then B's statement would be true and A's false, as required, and that if B were a knave, then B's statement would be false and A's false too, as required. Since both possibilities are consistent with the given information, we cannot determine whether B is a knight or a knave. ■

3. . . . seven inhabitants, A, B, C, D, E, F, and G, who tell you the following.

A: B is a knave or C is a knight.
B: B and E are both knights or both knaves.
C: B and F are both knights.
D: A and C are both knaves.
E: C is a knave.
F: Exactly one of B and G is a knight.
G: One of A and B is a knave and the other is a knight.

Determine, if you can, whether each of A–G is a knight or a knave. *[4]*

Solution. We can determine the types of three of the seven:

C	Knave
E	Knight
G	Knight

The rest could be knights or knaves, but there are only two sets of possibilities:

$$\text{either} \quad \left\{ \begin{array}{cc} \text{A} & \text{Knight} \\ \text{B} & \text{Knave} \\ \text{D} & \text{Knave} \\ \text{F} & \text{Knight} \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{cc} \text{A} & \text{Knave} \\ \text{B} & \text{Knight} \\ \text{D} & \text{Knight} \\ \text{F} & \text{Knave} \end{array} \right\}.$$

Here's the reasoning:

First, C must be a knave. For if C were a knight, C's statement would have to be true, so B and F would both be knights. However, then B's statement must be true and, since B is already known to be a knight, E would have to be a knight as well. Since E asserts that C is a knave, not a knight, we have a contradiction. Thus C must be a knave.

Second, since C is a knave, E's statement is true, and so E must be a knight.

Third, A and B must be of different types. Since C must be a knave, A's statement, "B is a knave or C is a knight," is true – and so A is a knight – exactly when B is a knave. (This is where we need to interpret "or" inclusively, so that "B is a knave or C is a knight" is true when either one or both of "B is a knave" or "C is a knight" is true.) Similarly, A's statement, "B is a knave or C is a knight," is false – and so A is a knave – exactly when B is a not a knave, that is, B is a knight.

Fourth, since A and B must be of different types, G's statement is true, and hence G is a knight.

Fifth, A and D must be of different types. Since C must be a knave, D's statement, "A and C are both knaves," would be true – and hence D would be a knight – if A is a knave. Similarly, D's statement would be false – and hence D would be a knave – if A is a knight. Note that the fact that both B and D must be of a different type from A means that B must be of the same type as D.

Sixth, F must be of the opposite type from B. Since G must be a knight, F's statement, "Exactly one of B and G is a knight," would be true – and hence F would be a knight – if B is a knave, and it will be false – and hence F would be a knave – if B is a knight. Note that the fact that F must be of the opposite type from B means that F must be of the same type as A.

We can thus determine the types of C, E, and G, while there are two sets of possibilities for the types of A, B, D, and F. The paranoid – er, careful – can double-check that the truth of each statement works out properly in each of the two possibilities. ■