Mathematics 235H – Linear algebra II: Vector spaces
Trent University, Winter 2008

Final Examination
Tuesday, 15 April, 2008

Time: 3 hours *Brought to you by* Stefan Bilaniuk.

Instructions: Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; annotated *Formula for Success* or an $8.5'' \times 11''$ aid sheet; one brain.

Part I. Do all of **1**–**5**.

1. Determine whether $W = \left\{ \mathbf{A} \in M_{33}(\mathbb{R}) \mid \mathbf{A}^T = \mathbf{A} \right\}$ is a subspace of $M_{33}(\mathbb{R})$. *[10]*

2. Verify that $\mathcal{B} = \left\{ x^2 + x, x^2 + 1, x + 1 \right\}$ is a basis for $\mathcal{P}_2$. *[10]*

3. Let $T : M_{22}(\mathbb{R}) \to \mathbb{R}$ be defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$.

a. Verify that T is a linear transformation. *[5]*

b. Find the rank and nullity of T. *[10]*

4. Determine whether $\langle \mathbf{A}, \mathbf{B} \rangle = |\mathbf{AB}| = \det(\mathbf{AB})$ is an inner product on $M_{44}(\mathbb{R})$. *[10]*

5. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ are both bases for $\mathbb{R}^2$. Find the change-of-basis matrices

a. $P_{\mathcal{C} \leftarrow \mathcal{B}}$ (from $\mathcal{B}$ to $\mathcal{C}$) *[10]*, and

b. $P_{\mathcal{B} \leftarrow \mathcal{C}}$ (from $\mathcal{C}$ to $\mathcal{B}$). *[5]*

(If you wish, you can think of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ as the matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^2$.)

Part II. Do any *three* of **6**–**11**.

6. Suppose U and W are both subspaces of a vector space V. Verify that $U \cap W = \{ \mathbf{v} \in V \mid \mathbf{v} \in U \text{ and } \mathbf{v} \in W \}$ is also a subspace of V. *[10]*

7. Show that $\|ax + b\| = \max(|2a|, |b|)$ defines a norm on $\mathcal{P}_1$. *[10]*

8. Suppose $S : U \to V$ and $T : V \to W$ are linear transformations between the vector spaces U, V, and W. Show the following:

a. $\ker(S) \subseteq \ker(T \circ S)$ *[5]*

b. $\operatorname{ran}(T \circ S) \subseteq \operatorname{ran}(T)$ *[5]*

9. Suppose V is a vector space, $\mathbf{v}$ a vector in V, and c a scalar. Show that if $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$. *[10]*

10. Suppose V is a vector space with an inner product $\langle\, , \,\rangle$, from which we define a norm $\|\ \|$ and distance function d in the usual way, and suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in V. Show that $d(\mathbf{u}, \mathbf{v}) = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}$ if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. *[10]*

11. Find the least squares approximating line $y = mx + b$ for the data points $(1, 0)$, $(2, 1)$, and $(3, 5)$, and compute the corresponding least squares error. *[10]*

[Total = 90]

Part □. Bonus!

△. Write an original little poem about linear algebra or mathematics in general. *[2]*

I HOPE THAT YOU ENJOYED THE COURSE.
HAVE A GOOD SUMMER!!