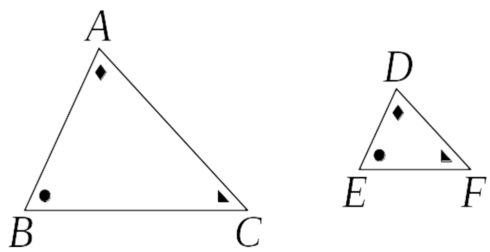
Similar Triangles and Similarity Criteria

Informally, two triangles are said to be *similar* if they have the same shape, though not necessarily the same size. Officially:

> Definition. $\triangle ABC$ is *similar* to $\triangle DEF$, often written as $\triangle ABC \sim \triangle DEF$, if corresponding angles of the triangles are equal, *i.e.* $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$.



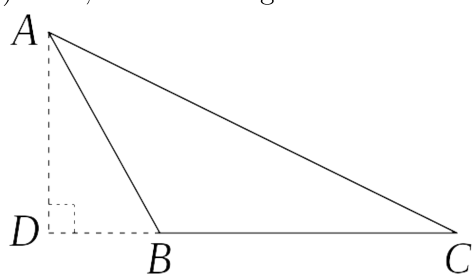
Similarity is a very useful tool that we will get into a little earlier than Euclid does in the *Elements* by developing some of the basic properties of similar triangles from high-school trigonometry. To make sure trigonometry works as intended, we will assume thoughout that the sum of the interior angles of a triangle is equal to two right angles (or π radians, or 180°). (This fact turns out to be equivalent to Euclid's parallel postulate, as we'll see later on.)

We will first prove the Law of Sines and the Law of Cosines, both of which we will use later to help prove things about similarity of triangles.

The Law of Sines. In any $\triangle ABC$:

$$\frac{\sin(\angle BAC)}{|BC|} = \frac{\sin(\angle ABC)}{|AC|} = \frac{\sin(\angle ACB)}{|AB|}$$

Proof. Suppose we are given $\triangle ABC$. Draw a line perpendicular to BC from A, meeting (an extension of) BC (if necessary) at D, as in the diagram below.



Note that $\triangle ADB$ and $\triangle ADC$ are right triangles with $\angle ADC = \triangle ADB$ as the right angle(s). Note further that if $\angle ABC$ is obtuse, as in the given diagram, then, measuring angles in radians, $\angle ABD = \pi - \angle ABC$ and so $\sin(\angle ABC) = \sin(\pi - \angle ABC) = \sin(\angle ABD)$. If $\angle ABC$ were to be acute instead, then $\angle ABD = \angle ABC$ and so we would still have $\sin(\angle ABC) = \sin(\angle ABD)$. (We leave it to the interested reader to figure out what happens when $\angle ABC$ is a right angle, in which case $B = D$. :-)

It follows that

$$\sin(\angle ACB) = \sin(\angle ACD) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{|AD|}{|AC|}$$
$$\sin(\angle ABC) = \sin(\angle ABD) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{|AD|}{|AB|}$$

Thus $|AC|\sin(\angle ACB) = |AD| = |AB|\sin(\angle ABC)$. Dividing on both ends of the equation by $|AC| \cdot |AB|$ now gives us $\dfrac{\sin(\angle ABC)}{|AC|} = \dfrac{\sin(\angle ACB)}{|AB|}$. A similar argument will show that $\dfrac{\sin(\angle BAC)}{|BC|} = \dfrac{\sin(\angle ABC)}{|AC|}$, completing the proof of the Law of Sines. $\square$

THE LAW OF COSINES. In any $\triangle ABC$:

$$|BC|^2 = |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle BAC)$$

PROOF. I really can't improve on the presentation of the standard trigonometric proof from the Wikipedia article about the Law of Cosines:

Using trigonometry [edit]

Dropping the perpendicular onto the side c through point C, an altitude of the triangle, shows (see Fig. 5)

$$c = a\cos\beta + b\cos\alpha.$$

(This is still true if α or β is obtuse, in which case the perpendicular falls outside the triangle.) Multiplying through by c yields

$$c^2 = ac\cos\beta + bc\cos\alpha.$$

Considering the two other altitudes of the triangle yields

$$a^2 = ac\cos\beta + ab\cos\gamma,$$
$$b^2 = bc\cos\alpha + ab\cos\gamma.$$

Adding the latter two equations gives

$$a^2 + b^2 = ac\cos\beta + bc\cos\alpha + 2ab\cos\gamma.$$

Subtracting the first equation from the last one results in

$$a^2 + b^2 - c^2 = ac\cos\beta + bc\cos\alpha + 2ab\cos\gamma - (ac\cos\beta + bc\cos\alpha)$$

which simplifies to

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

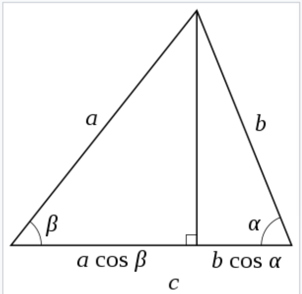


Fig. 5 – An acute triangle with perpendicular

This proof uses trigonometry in that it treats the cosines of the various angles as quantities in their own right. It uses the fact that the cosine of an angle expresses the relation between the two sides enclosing that angle in *any* right triangle. Other proofs (below) are more geometric in that they treat an expression such as $a\cos\gamma$ merely as a label for the length of a certain line segment.

$\square$

Note that the Law of Cosines reduces to the Pythagorean Theorem when $\angle BAC$ is a right angle.

THEOREM. Several useful facts about similarity of triangles are equivalent to the definition:

$$\begin{aligned}
\triangle ABC \sim \triangle DEF \iff & \angle ABC = \angle DEF,\ \angle BCA = \angle EFD, \text{ and } \angle CAB = \angle FDE \\
& \text{[Definition]} \\
\iff & \angle ABC = \angle DEF \text{ and } \angle BCA = \angle EFD \\
& \text{[Angle-Angle Similarity Criterion]} \\
\iff & \frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|} \\
& \text{[Side-Side-Side Similarity Criterion]} \\
\iff & \angle CAB = \angle FDE \text{ and } \frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} \\
& \text{[Side-Angle-Side Similarity Criterion]}
\end{aligned}$$

PROOF. *1. Similarity implies the Angle-Angle Similarity Criterion.* This is trivial: if $\triangle ABC \sim \triangle DEF$, so $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$, then we certainly have $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$.

2. The Angle-Angle Similarity Criterion implies Similarity. Suppose $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$. Since the sum of the internal angles of any triangle is two right angles, we also have

$$\angle CAB = \pi - \angle ABC - \angle BCA = \pi - \angle DEF - \angle EFD = \angle FDE.$$

Since all three pairs of corresponding angles are equal, $\triangle ABC \sim \triangle DEF$ by definition.

3. Similarity implies the Side-Side-Side Similarity Criterion. Suppose $\triangle ABC \sim \triangle DEF$, so $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$. By the Law of Sines we have

$$\begin{aligned} \frac{\sin(\angle BAC)}{|BC|} &= \frac{\sin(\angle ABC)}{|AC|} = \frac{\sin(\angle ACB)}{|AB|} \\ \text{and } \frac{\sin(\angle EDF)}{|EF|} &= \frac{\sin(\angle DEF)}{|DF|} = \frac{\sin(\angle DFE)}{|DE|}\,. \end{aligned}$$

Inverting the former equation and multiplying it into the latter, we get

$$\frac{\sin(\angle EDF)}{|EF|} \cdot \frac{|BC|}{\sin(\angle BAC)} = \frac{\sin(\angle DEF)}{|DF|} \cdot \frac{|AC|}{\sin(\angle ABC)} = \frac{\sin(\angle DFE)}{|DE|} \cdot \frac{|AB|}{\sin(\angle ACB)}\,.$$

Because the corresponding angles are equal, they cancel out, leaving us with

$$\frac{|BC|}{|EF|} = \frac{|AC|}{|DF|} = \frac{|AB|}{|DE|}\,,$$

as required.

4. The Side-Side-Side Similarity Criterion implies Similarity. Suppose that $\frac{|BC|}{|EF|} = \frac{|AC|}{|DF|} = \frac{|AB|}{|DE|}$. Let $r = \frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$. By the Law of Cosines, we have

$$\begin{aligned} |BC|^2 &= |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle BAC) \\ \text{and } |EF|^2 &= |DE|^2 + |DF|^2 - 2 \cdot |DE| \cdot |DF| \cdot \cos(\angle EDF)\,. \end{aligned}$$

Multiplying the latter equation by r^2 yields:

$$\begin{aligned} & r^2|EF|^2 = r^2|DE|^2 + r^2|DF|^2 - 2r^2 \cdot |DE| \cdot |DF| \cdot \cos(\angle EDF) \\ \Longrightarrow\ & \left(\frac{|BC|}{|EF|}\right)^2 |EF|^2 = \left(\frac{|AB|}{|DE|}\right)^2 |DE|^2 + \left(\frac{|AC|}{|DF|}\right)^2 |DF|^2 \\ & \qquad - 2 \cdot \frac{|AB|}{|DE|} \cdot \frac{|AC|}{|DF|} \cdot |DE| \cdot |DF| \cdot \cos(\angle EDF) \\ \Longrightarrow\ & |BC|^2 = |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle EDF) \end{aligned}$$

It follows that

$$\cos(\angle BAC) = \frac{|BC|^2 - |AB|^2 - |AC|^2}{-2 \cdot |AB| \cdot |AC|} = \cos(\angle EDF)\,.$$

Since the interior angles in a triangle are each between 0 and two right angles and the cosine function is 1–1 on this domain, it follows that $\angle BAC = \angle EDF$. Similar arguments can be used to show that $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$, so the triangles are similar, *i.e.* $\triangle ABC \sim \triangle DEF$, by definition.

5. Similarity imples the Side-Angle-Side Similarity Criterion. Suppose $\triangle ABC \sim \triangle DEF$, so $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$. By part *3* above, it follows that $\frac{|BC|}{|EF|} = \frac{|AC|}{|DF|} = \frac{|AB|}{|DE|}$. In particular, we have $\angle CAB = \angle FDE$ and $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$, the conditions for the Side-Angle-Side Similarity Criterion.

6. The Side-Angle-Side Similarity Criterion implies Similarity. Suppose $\angle CAB = \angle FDE$ and $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$. Let $r = \frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$. By the Law of Cosines, we have

$$\begin{aligned} |BC|^2 &= |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle CAB) \\ \text{and } |EF|^2 &= |DE|^2 + |DF|^2 - 2 \cdot |DE| \cdot |DF| \cdot \cos(\angle FDE) \, . \end{aligned}$$

Multiplying the latter equation by r^2 yields and using $\angle CAB = \angle FDE$:

$$\begin{aligned} r^2|EF|^2 &= r^2|DE|^2 + r^2|DF|^2 - 2r^2 \cdot |DE| \cdot |DF| \cdot \cos(\angle FDE) \\ &= \left(\frac{|AB|}{|DE|}\right)^2 |DE|^2 + \left(\frac{|AC|}{|DF|}\right)^2 |DF|^2 \\ &\qquad - 2 \cdot \frac{|AB|}{|DE|} \cdot \frac{|AC|}{|DF|} \cdot |DE| \cdot |DF| \cdot \cos(\angle CAB) \\ &= |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle CAB) \\ &= |BC|^2 \end{aligned}$$

It follows that $r^2 = \frac{|BC|^2}{|EF|^2}$, and so $r = \frac{|BC|}{|EF|} = \frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$. It now follows from the Side-Side-Side Similarity Criterion (see part *4* above) that the triangles are similar, *i.e.* $\triangle ABC \sim \triangle DEF$.

This completes the proof. Whew! □

Note that similarity of triangles works somewhat differently in geometries where the parallel postulate fails: in both hyperbolic [many lines through a given point that are parallel to a given line] and elliptic geometries [no parallel lines at all], similarity of triangles is the same as congruence, *i.e.* triangles that have the same shape also have the same size.