Mathematics 2260H – Geometry I: Euclidean Geometry
Trent University, Winter 2025

Take-Home Final Examination
Due on Thursday, 17 April.*

Instructions: Do both of parts **I** and **II**, and, if you wish, part **III** as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part I. Do all *four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*
Please draw any relevant diagram(s) in each problem that you choose to do!

1. Given a circle, show that there is a square whose corners are on the circle, using only Euclid's Postulates (plus Postulates A and S) and the Propositions in Book I of the *Elements*.

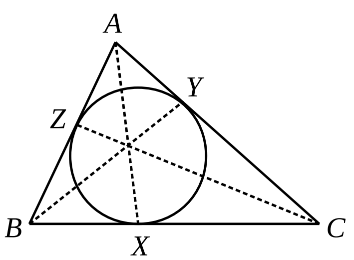
2. Suppose two circles of equal radius are tangent to each other, and that lines ℓ and m, which are different from each other, are each tangent to both circles. Explain why ℓ and m are either parallel or perpendicular to each other.

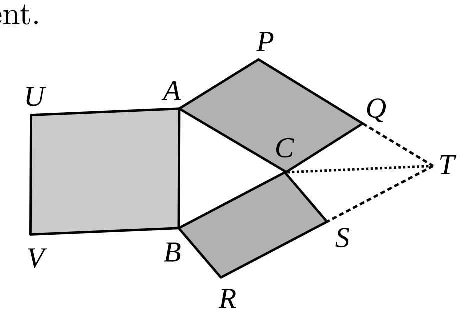
3. Given a line segment AB show, using only Euclid's Postulates (plus Postulates A and S) and the Propositions in Book I of the *Elements*, how to construct:
a. A line segment AC such that $|AC| = 3|AB|$. *[5]*
b. A line segment AD such that $|AB| = 3|AD|$. *[5]*

4. Suppose $\triangle ABC$ is isosceles, with $|AB| = |AC|$, and D is a point on the same side of BC as A such that $\angle BAC = 2\angle DBC$. Show that A is the circumcentre of $\triangle DBC$.

Part II. Do any *four* (4) of problems **6** – **12**. *[40 = 4 × 10 each]*
Please draw any relevant diagram(s) in each problem that you choose to do!

5. Suppose that the incircle of $\triangle ABC$ touches AB at Z, BC at X, and AC at Y. Show that AX, BY, and CZ are concurrent.





[The rest of Part **II** *and Part* **III** *are on page 2.]*

* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.

*[Part **I** and the rest of Part **II** are on page 1.]*

6. Suppose we are given $\triangle ABC$ and parallelograms $ACQP$ and $BCSR$. Let T be the point where PQ intersects RS. Connect C to T, and let $ABVU$ be the paralellogram such that $AU \parallel TC \parallel BV$ and $|AU| = |TC| = |BV|$. Show that the area of $ABVU$ is equal to the sum of the areas of $ACQP$ and $BCSR$. *[Another theorem of Pappus'.]*

7. A polygon is said to be *convex* if the line segments joining two vertices of the polygon never pass outside the polygon.

- **a.** Show that the sum of the interior angles of a convex n-sided polygon is $(n-2)$ straight angles. *[5]*
- **b.** Show that the sum of the interior angles of an arbitrary n-sided polygon is $(n-2)$ straight angles. *[5]*

8. A geometry $\mathcal{G}$ is defined as follows:

- The points of $\mathcal{G}$ are the points of the Cartesian plane inside the unit circle centred at the origin, *i.e.* $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 < 1$.
- The lines of $\mathcal{G}$ are the chords of the unit circle centred at the origin. (Not including the endpoints because points that are actually on the unit circle are not in $\mathcal{G}$.)
- Intersections, angles, distances, *etc.*, work as is usual in the Cartesian plane.

Determine, with proof, which of Euclid's five Postulates are true in $\mathcal{G}$.

9. Suppose $ABCD$ is a parallelogram whose diagonals AC and BD are perpendicular to each other. Show that the sides of the parallelogram are all of equal length.

10. Suppose the radius of the incircle of $\triangle ABC$ is r and the *semiperimeter* of the triangle is $s = \frac{1}{2}(|AB| + |BC| + |CA|)$. Show that the area of the triangle is equal to rs.

11. Given a rectangle $ABCD$, show that there is a circle passing through the points A and B which is also tangent to the side CD.

12. Determine, with proof, the maximum number of points which are all the same distance from each other that can be found in the Euclidean plane. How does the answer change if the points are in $\mathbb{R}^3$ instead?

[Total = 80]

Part III. Bonus!

$\triangle$. Write an original poem about Euclidean geometry. *[1]*

$\square$. A curve consists of all of the points (x, y) in the Cartesian plane such that the sum of the distances from (x, y) to $(2, 0)$ and to $(-2, 0)$ is 8. Find an equation for the curve that does not employ square roots. *[1]*

I hope it's been fun!
Enjoy the summer!