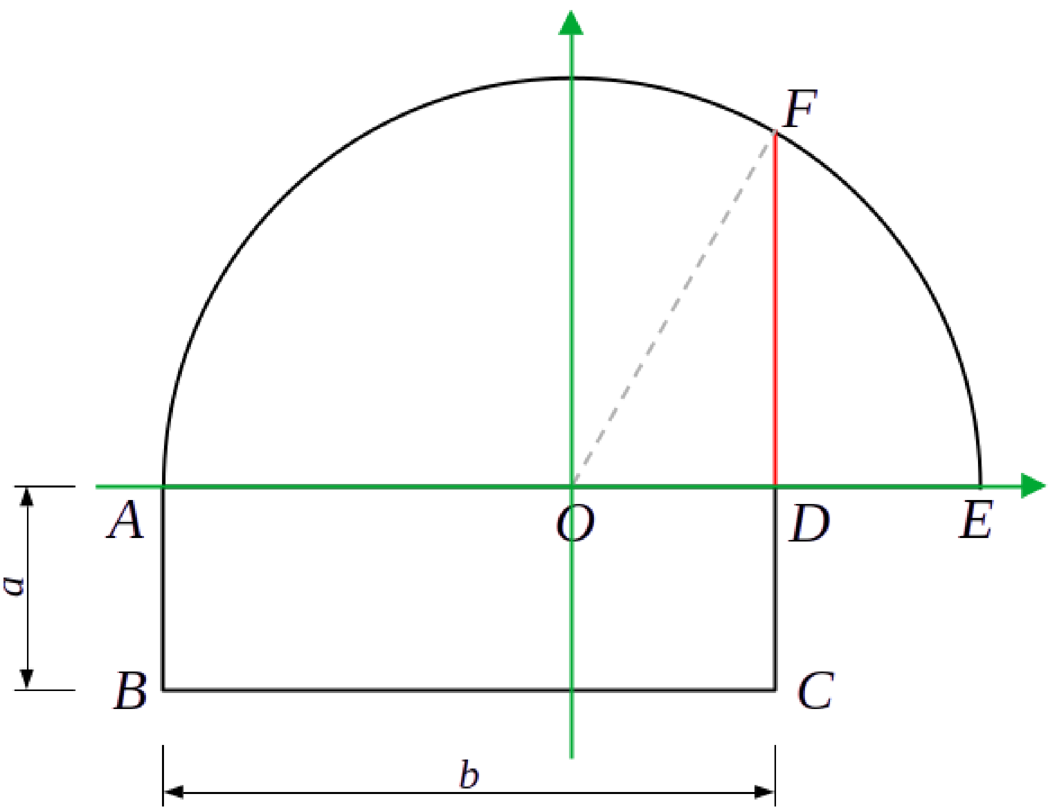
Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Winter 2025

Solution to Assignment #7
Squaring a Rectangle

On Assignment #5 you were asked to construct a square equal in area to a rectangle that had 2 to 1 proportions, which made the task fairly easy. Here is a method that works for any rectangle:

> Suppose we are given rectangle $ABCD$, with the vertices listed in clockwise order from the top left. We can assume that $|AD| > |AB|$. (Why?) Exrend AD past D to E so that $|DE| = |AB|$. Let O be the midpoint of AE. Draw the circle with centre O and radius OE. Extend CD past D until it meets the circle at F. Then a square constructed on DF will have area equal to that of the rectangle $ABCD$.
>
> The diagram below omits the final square to avoid clutter. *It has been modified per the given hint to place it in the Cartesian plane.*



1. Show, using any method you like, that this method works. *[10]*

Hint: The easiest method known to your instructor starts with placing the diagram so that O is at the origin in the Cartesian plane and AE lies along the x-axis. Work out the y-coordinate of F ... For convenience in doing the algebra, let $|AB| = a$ and $|AD| = b$. If you want to do things the hard way, this is most of Proposition II-14 in Euclid's *Elements*.

SOLUTION. Following the hint, let $a = |AB|$ and $b = |AD|$ and place the diagram so that O is at the origin and AE lies along the x-axis, as in the modified diagram above.

Since $|AE| = |AD| + |DE| = |AD| + |AB| = b + a$ and $O = (0,0)$ is the midpoint of AE, it is easy to see that $A = \left(-\frac{b+a}{2}, 0\right)$ and $E = \left(\frac{b+a}{2}, 0\right)$. As $|DE| = a$ and D is to the left E, it follows that $D = \left(\frac{b+a}{2} - a, 0\right) = \left(\frac{b-a}{2}, 0\right)$. Also, since OE is a radius of the circle, which is centred at the origin, the circle has the Cartesian equation $x^2 + y^2 = \left(\frac{b+a}{2}\right)^2$.

Observe that $|FD|$ is the y-coordinate of F, which has the same x-coordinate as D, namely $\frac{b-a}{2}$. We plug this x value into the equation of the circle and solve for the y-value.

$$\begin{aligned}
\left(\frac{b-a}{2}\right)^2 + y^2 = \left(\frac{b+a}{2}\right)^2 \implies y^2 &= \left(\frac{b+a}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 \\
&= \frac{b^2 + 2ab + a^2}{4} - \frac{b^2 - 2ab + a^2}{4} \\
&= \frac{2ab - (-2ab)}{4} = \frac{4ab}{4} = ab
\end{aligned}$$

As $y^2 = ab$ and we don't have negative lengths, it follows that $|FD| = y = \sqrt{ab}$.

A square with FD as one of its sides will therefore have area $\left(\sqrt{ab}\right)^2 = ab$, which is the area of the rectangle $ABCD$, as desired. ■