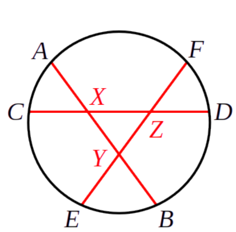
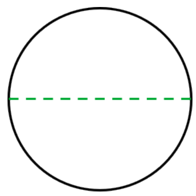
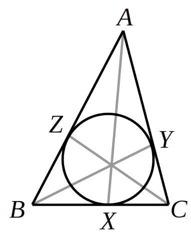
Mathematics 2260H – Geometry I: Euclidean Geometry
Trent University, Winter 2024

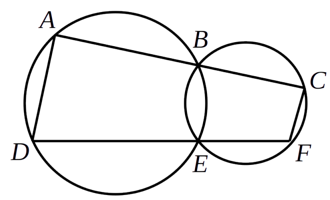
Take-Home Final Examination
Due on Thursday, 18 April.*

Instructions: Do both of parts **I** and **II**, and, if you wish, part **III** as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part I. Do any *four* (4) of problems **1** – **5**. *[40 = 4 × 10 each]*





1. Suppose AB, CD, and EF are chords of a circle, X is the intersection of AB and CD, Y is the intersection of AB and EF, and Z is the intersection of CD and EF, as in the diagram above. Show that if $|AX| = |CX| = |EY| = |BY| = |DZ| = |FZ|$, then $\triangle XYZ$ is equilateral.

2. Use Euclid's Postulates (plus Postulates A and S) and the Propositions in Euclid's *Elements* to construct a diameter of a given circle, whose centre is not given.

3. Suppose X, Y, and Z are the points on BC, AC, and AB, respectively, where the incircle of $\triangle ABC$ touches the sides. Show that the cevians AX, BY, and CZ are concurrent.

4. Two circles intersect at points B and E. Suppose that A and D are points on the first circle and C and F are points on the second circle such that A, B, and C are collinear and D, E, and F are collinear. Show that AD is parallel to CF.

5. Find a finite set of tiles that are regular polygons, including a regular dodecagon (*i.e.* 12-gon), that can be used to tile the plane periodically.

*[Parts **II** and **III** are on page 2.]*

* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.

*[Part **I** is on page 1.]*

Part II. Do any *four* (4) of problems **6** – **12**. *[40 = 4 × 10 each]*
Please draw any relevant diagram(s) in each problem that you choose to do!

6. Suppose that the centre N of the nine-point circle and orthocentre H of $\triangle ABC$ happen to be the same point. Show that $\triangle ABC$ is equilateral.

7. Suppose A, B, and C are distinct points on a line ℓ, and A', B', and C' are distinct points not on ℓ such that the points $D = AB' \cap A'B$, $E = AC' \cap A'C$, and $F = BC' \cap B'C$ exist and are collinear. Show that A', B', and C' are also collinear. *[A converse of sorts to Desargues' Theorem.]*

8. Suppose $ABCD$ is a parallelogram whose diagonals AC and BD are perpendicular to each other. Show that the sides of the parallelogram are all of equal length.

9. Given a rectangle $ABCD$ with $|AD| = |BC| = 2|AB| = 2|CD|$, give a straightedge-and-compass construction of a square equal in area to the given rectangle.

10. Suppose P, Q, and R are the midpoints of the sides BC, AC, and AB of $\triangle ABC$, respectively, and that O is the circumcentre of the triangle. Let X, Y, and Z be the points on OP, OQ, and OR, respectively, that are across on the other side of P, Q, and R, respectively, from O, and such that $|OP| = |PX|$, $|OQ| = |QY|$, and $|OR| = |RZ|$. Show that $\triangle XYZ \cong \triangle ABC$.

11. Suppose the points of a geometry $\mathcal{G}$ are the points strictly inside the unit circle centred at the origin in $\mathbb{R}^2$, *i.e.* (x, y) such that $x^2 + y^2 < 1$, and the lines of the geometry are the chords of the circle. Angles and distances are measured just as in $\mathbb{R}^2$. Determine which of Euclid's five postulates hold in $\mathcal{G}$.

12. Suppose the radius of the incircle of $\triangle ABC$ is r and the *semiperimeter* of the triangle is $s = \frac{1}{2}(|AB| + |BC| + |CA|)$. Show that the area of the triangle is equal to rs.

[Total = 80]

Part III. Bonus!

△. Write an original poem about Euclidean geometry. *[1]*

□. A curve consists of all of the points (x, y) in the Cartesian plane such that the sum of the distances from (x, y) to $(1, 0)$ and to $(-1, 0)$ is 4. Find an equation for the curve that does not employ square roots. *[1]*

It's been fun!
Enjoy the summer!