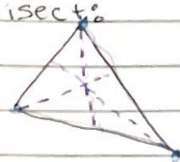
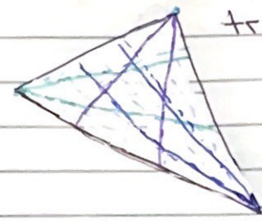
March 21, 2024

Morley's Trisector Theorem (1899)

bisect:



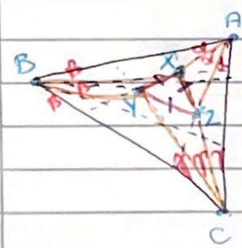
trisect:



In any triangle, the angle trisectors adjacent to each side intersect in the vertices of an equilateral triangle

Proof: (John Conway, 1995)

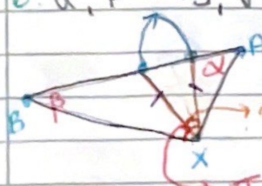
$\triangle ABC$ Let X be the intersection of the angle trisectors adjacent to AB, Y... to BC & Z... to AC



Let α, β, γ be such that
$\angle BAC = 3\alpha$
$\angle ABC = 3\beta$
& $\angle ACB = 3\gamma$

We can construct triangle with angle

0. $\pi/3, \pi/3, \pi/3$ scaled to have sidelength 1
1. $\alpha, \beta, \pi - \alpha - \beta$ scaled to have $|XP| = |XQ| = 1$
2. $\beta, \gamma, \pi - \beta - \gamma$ } Similarly here...
3. $\gamma, \alpha, \pi - \gamma - \alpha$ }
4. $\beta, \alpha + \pi/3, \gamma + \pi/3$ } scaled to sidelength
5. $\gamma, \alpha + \pi/3, \beta + \pi/3$ } opposite β, γ, α has
6. $\alpha, \beta + \pi/3, \gamma + \pi/3$ } length 1

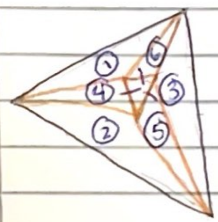


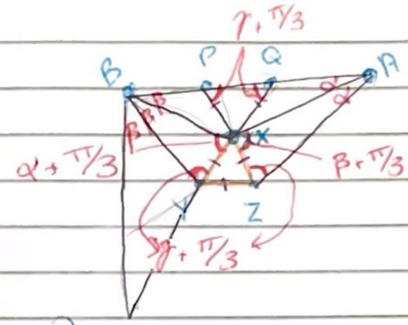
$2\pi/3 - \gamma + \alpha$...
$\gamma + \pi/3$

$\gamma + 2\pi/3$

$\pi - \alpha - \beta$

$\pi - (\pi/3 - \gamma)$
$= 2\pi/3 + \gamma$





$\triangle AXP \cong \triangle AXZ$ } So the sidelengths match
$\triangle BXQ \cong \triangle BXY$ } where glued together

We can do similar things for $\triangle BYC$ & $\triangle AZC$ so all the pieces fit together since all the angles at the verticies are the same the assembled triangle is similar to the original.

So the original intersectioning make an equilateral triangle //