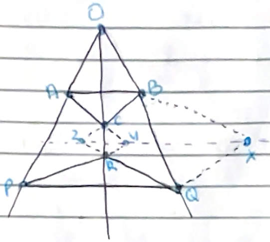
Lecture 28: Desargues' theorem

March 21, 2024



Desargues' theorem:

Given triangles $\triangle ABC$ & $\triangle PQR$, PA, QB, RC are concurrent if and only if:

$X = AB \cap PQ$,

$Y = AC \cap PR$,

& $Z = BC \cap QR$ are collinear.

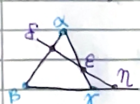
informal "proof"

If we're in 3D & $\triangle ABC$ & $\triangle PQR$ are in perspective from O & define different planes we get.

Proof: we'll use Menelaus' theorem

$\boxed{\Rightarrow}$ Assume that there is a point O such that PA, QB, & RC are concurrent in O,

by Menelaus' theorem.



$\delta, \varepsilon, \eta$ are collinear iff $\dfrac{\alpha\delta}{\delta\beta} \cdot \dfrac{\beta\eta}{\eta\gamma} \cdot \dfrac{\gamma\varepsilon}{\varepsilon\alpha} = -1$

It will suffice to show that

$$\frac{AX}{XB} \cdot \frac{BZ}{ZC} \cdot \frac{CY}{YA} = -1$$

$\triangle OAC$: P, R, Y are collinear, so

$$\frac{OP}{PA} \cdot \frac{AY}{YC} \cdot \frac{CR}{RO} = -1 \quad (1)$$

$\triangle OCB$: Q, R, Z are Collinear, so

$$\frac{OQ}{QB} \cdot \frac{BZ}{ZC} \cdot \frac{CR}{RO} = -1 \quad ②$$

$\triangle OAB$: P, Q, X are collinear, so

$$\frac{OP}{PA} \cdot \frac{AX}{XB} \cdot \frac{BQ}{QO} = -1 \quad ③$$

Then $\frac{② \cdot ③}{①}$

$$= \frac{(-1)\cdot(-1)}{(-1)} = -1$$

$$= \frac{\frac{\cancel{OQ}}{\cancel{QB}} \cdot \frac{BZ}{ZC} \cdot \frac{\cancel{CR}}{\cancel{RO}} \cdot \frac{\cancel{OP}}{\cancel{PA}} \cdot \frac{AX}{XB} \cdot \frac{\cancel{BQ}}{\cancel{QO}}}{\frac{\cancel{OP}}{\cancel{PA}} \cdot \frac{AY}{YC} \cdot \frac{\cancel{CR}}{\cancel{RO}}}$$

$$= \frac{BZ}{ZC} \cdot \frac{AX}{XB} \cdot \frac{YC}{AY}$$

$\therefore$ X, Y, Z are Collinear.

$\boxed{\Leftarrow}$ Assume X, Y, Z are Collinear.

[To Show: PA, QB, & RC are concurrent]

Let $O = PA \cap QB$

[To Show: R, C, O are Collinear]

$\triangle APX$: O, B, Q are collinear, so

$$\frac{AB}{BX} \cdot \frac{XQ}{QP} \cdot \frac{PO}{OA} = -1 \quad ①$$

To show:

$\triangle APY$: C, R, O are collinear iff:

$$\frac{AC}{CY} \cdot \frac{YR}{RP} \cdot \frac{PO}{OA} = -1$$

$\triangle PXY$ R, Q, Z are collinear, so

$$\frac{PR}{RY} \cdot \frac{YZ}{ZX} \cdot \frac{XQ}{QP} = -1 \quad ②$$

Lecture 28: proof continued

March 21, 2024

$\triangle AYX$: Z, C, B are Collinear, So:

$$\frac{AB}{BX}\cdot\frac{XZ}{ZY}\cdot\frac{YC}{CA} = -1 \qquad (3)$$

Now: $\dfrac{(1)}{(2)\cdot(3)} = \dfrac{(-1)}{(-1)\cdot(-1)} = -1$

$$= \frac{\cancel{\frac{AB}{BX}}\cdot\cancel{\frac{XQ}{QP}}\cdot\frac{PO}{OA}}{\frac{PR}{RY}\cdot\cancel{\frac{YZ}{ZX}}\cdot\cancel{\frac{XQ}{QA}}\cdot\cancel{\frac{AB}{BX}}\cdot\cancel{\frac{XZ}{ZY}}\cdot\frac{YC}{CA}}$$

$$= \frac{CA}{YC}\cdot\frac{PO}{OA}\cdot\frac{RY}{PR}$$

$$= \frac{AC}{CY}\cdot\frac{YR}{RP}\cdot\frac{PO}{OA} = -1 \quad //$$

∴ Collinear