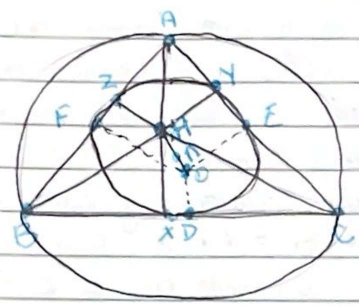
March 12, 2024

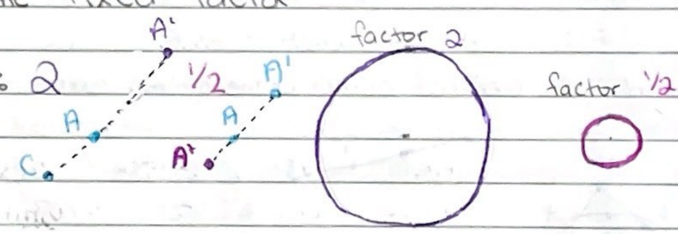
Lecture 24: more on the nine-point Circle

Proposition: The centre of the nine-point Circle, N, is on the Euler line halfway between the orthocentre H and the circumcentre O.



Definition: A dilation of the plane about point "P" moves every point plane to (or away) by some fixed factor

Factor: 2



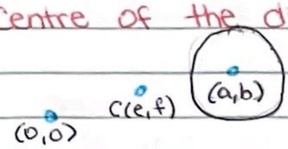
Lemma: A dilation moves circles to circles

Proof: Identify the plane with $\mathbb{R}^2$, translate the centre plane so that the centre of the circle is at the orgin.

$x^2+y^2=r^2$ Do the dilation about the orgin

... Still a circle, move it back

What if the centre of the circle is not the centre of the dilation?



$(x,y) \to (c-e, y-f)$

(x,y) on the circle

$\Rightarrow (x-e, y-f)$

$(x-a-e)^2+(y-b-f)^2$

$((x-a)-e)^2+((y-b)-f)^2$

$=(x-a)^2-2e(x-a)+e^2+(y-b)^2-2f(y-b)+f^2$

$=r^2$ ... $\to$ is this still a circle?

Lecture 24: Proposition continued

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new circle: $(x-(a-e))^2 + (y-(b-f))^2 = r^2$

translate: $(e,f) \mapsto (0,0)$

dilate about $(0,0)$ by k

The points on the circle go to

$$(x,y) \mapsto (kx, ky)$$

Then the new points are on the circle

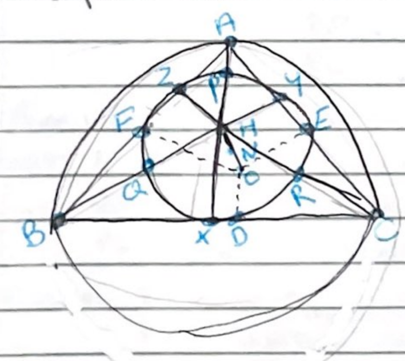
$$(kx-(ka-ke))^2 + (ky-(kb-kf))^2 = (kr)^2$$

Translating $(0,0) \mapsto (e,f)$ keeps this a circle with radius kr

Suppose $\triangle ABC$ has an orthocentre H & circumcentre O etc. as in diagram.

We'll do dilation with centre H and factor $1/2$. This will move the circumcentre O halfway to the orthocentre H.

Claim: This also moves the circumcentre to the nine-point circle



Consider: A, B, C on the circumcircle. These move to point P, Q, R resp. halfway between A, B, C resp. and H. But P, Q, R are on the nine-point circle. So O moves to it's centre N //

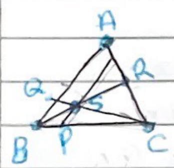
Cevian Definition: A Cevian is a line joining a vertex of a triangle to (an extension of) the opposite side.

Eg
- medians
- Altitudes

A convention given a line AB we pick a direction on it to be forwards. Then

$$AB = \begin{cases} |AB| & \text{if going from A to B is forward} \\ -|AB| & \text{if going from A to B is backward} \end{cases}$$

Ceva's theorem: Suppose AP, BQ and CR are Cevian's of $\triangle ABC$. Then AP, BQ, & CR are concurrent in a point S if and only if



$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1$$