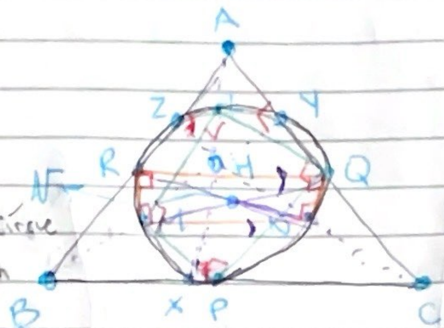
March 7, 2024

Lecture 23: The 9 point circle



Centre of a circle passing through R, Q, N, M

$\triangle ABC$:

X, Y, Z feet of altitudes from A, B, C resp.

P, Q, R midpoints of BC, AC, AB resp.

L, M, N midpoints of AH, BH, CH resp.

Theorem: All nine points are on the same circle

Proof:

We'll show that RQNM is a parallelogram

$RQ \parallel BC$ since R & Q are the midpoints of AB & AC ($\& |RQ| = \frac{1}{2}|BC|$)

& $MN \parallel BC$ since M and N are the midpoints of HB & HC ($\& |MN| = \frac{1}{2}|BC|$)

$\Rightarrow RQ \parallel MN$ & $|RQ| = |MN|$

In $\triangle AHB$, R is the midpoint of AB & M is the midpoint of BH. So $RM \parallel AH$ & $|RM| = \frac{1}{2}|AH|$.

In $\triangle AHC$, Q is the midpoint of AC & N is the midpoint of CH.

$\Rightarrow QN \parallel AH$ & $|QN| = \frac{1}{2}|AH|$

$\Rightarrow RM \parallel QN$ & $|RM| = |QN|$

Since AH is part of AX, it is perpendicular to BC & hence RQ & MN are perpendicular to RM & QN ie RQNM is a rectangle.

The diagonals RN & QM then intersect in a point N equidistant from R, Q, M, N & hence the centre of a circle passing through them.

A similar argument shows that PQLM is a rectangle

This shares a diagonal with RQNM so N is the midpoint of QM & hence also equidistant from all six of P, Q, L, M, R, N.

To show: X, Y, Z are also equidistant from N.
Draw the circle with centre N & passing P, Q, R, L, M, N.
LP is a diameter of the circle
$\angle LXP = \frac{\pi}{2}$, so by the converse of Thales' theorem, X is on the circle. Similarly, Y & Z are also on the circle. //

Proposition: The centre of a nine-point circle is on the Euler line.