Mathematics 2260H – Geometry I: Euclidean Geometry

TRENT UNIVERSITY, Winter 2023

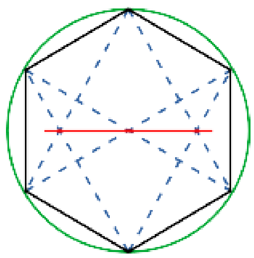
Take-Home Final Examination

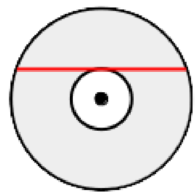
(Some typos corrected on 2023-04-05.)

Due on Friday, 21 April.

Instructions: Do both of parts ⬯ and $\triangle$, and, if you wish, part $\square$ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part ⬯. Do any *four* (4) of problems **1** – **5**. *[40 = 4 × 10 each]*

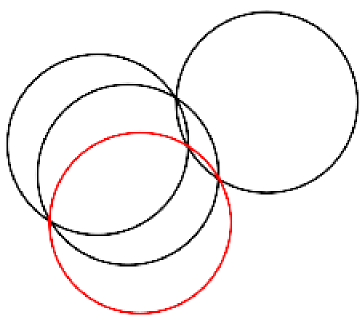


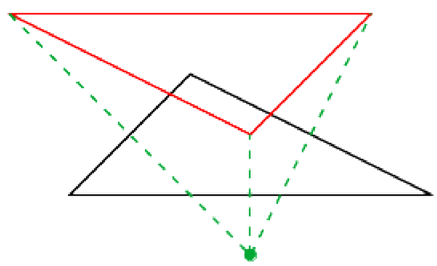


1. Suppose $ABCDEF$ is a regular hexagon, with the vertices in clockwise order. Let X be the intersection of AE and DF, Y be the intersection of BE and CF, and Z be the intersection of AC and BD. Show that X, Y, and Z are collinear.

2. Use Euclid's Postulates (plus Postulates A and S) and the Propositions in Book I of the *Elements* to show that a given line segment can be divided into three equal parts.

3. A chord AB of a circle with centre O is tangent to a smaller circle with centre O. Assuming that $|AB| = 12\ m$, determine the area of the annular region between the two circles.





4. Suppose that three circles of equal radius pass through a common point P, and denote by A, B, and C the three other points where two of these circles at a time intersect. Show that the circumcircle of $\triangle ABC$ has the same radius as the original three circles.

5. Suppose that O is the circumcentre of $\triangle ABC$ and points X, Y, and Z are chosen so that BC, AC, and AB are the perpendicular bisectors of OX, OY, and OZ, respectively. Show that $\triangle XYZ \cong \triangle ABC$.

[Parts $\triangle$ and $\square$ are on page 2.]

Part Δ. Do any *four* (4) of problems **6 – 12**. *[40 = 4 × 10 each]*
Please draw the relevant diagram(s) in each problem that you choose to do!

6. Suppose that the incentre and orthocentre of $\triangle ABC$ are the same point. Show that $\triangle ABC$ is equilateral.

7. Suppose A, B, and C are distinct points on a line ℓ, and A', B', and C' are distinct points not on ℓ such that the points $D = AB' \cap A'B$, $E = AC' \cap A'C$, and $F = BC' \cap B'C$ exist and are collinear. Show that A', B', and C' are also collinear. *[A converse of sorts of Desargues' Theorem.]*

8. Suppose we are given $\triangle ABC$ and parallelograms $ACQP$ and $BCSR$. Let T be the point where PQ intersects RS. Connect C to T, and let $ABVU$ be the paralellogram such that $AU \parallel TC \parallel BV$ and $|AU| = |TC| = |BV|$. Show that the area of $ABVU$ is equal to the sum of the areas of $ACQP$ and $BCSR$. *[A theorem of Pappus generalizing the Pythagorean Theorem.]*

9. Given a rectangle $ABCD$, give a straightedge-and-compass construction of a square equal in area to the given rectangle.

10. Suppose a smaller circle lies inside a larger circle and is tangent to the larger circle at T. Suppose P is a point outside the larger circle such that PT is tangent to the larger circle. Let S be the point on the smaller circle other than T such that PS is tangent to the smaller circle, and let Q and R be the intersections (of the extension) of PS with the larger circle. Show that $|PQ| \cdot |PR| = |PS| \cdot |PT|$.

11. Suppose AP, BQ, and CR are the angle bisectors of $\triangle ABC$, and suppose that S is a point on (an extension of) AB such that CS is perpendicular to CR. Show that P, Q, and S are collinear.

12. Suppose the radius of the incircle of $\triangle ABC$ is r and the *semiperimeter* of the triangle is $s = \frac{1}{2}(|AB| + |BC| + |CA|)$. Show that the area of the triangle is equal to rs.

[Total = 80]

Part $\square$. Bonus!

$\cong$. Write an original poem about Euclidean geometry. *[1]*

$\sim$. Give an example of triangles $\triangle ABC$ and $\triangle DEF$ which are *not* the same triangle, or even congruent, but which nevertheless have the same centroid G, orthocentre H, incentre I, and circumcentre O. *[1]*

It's been fun!
Enjoy the summer!