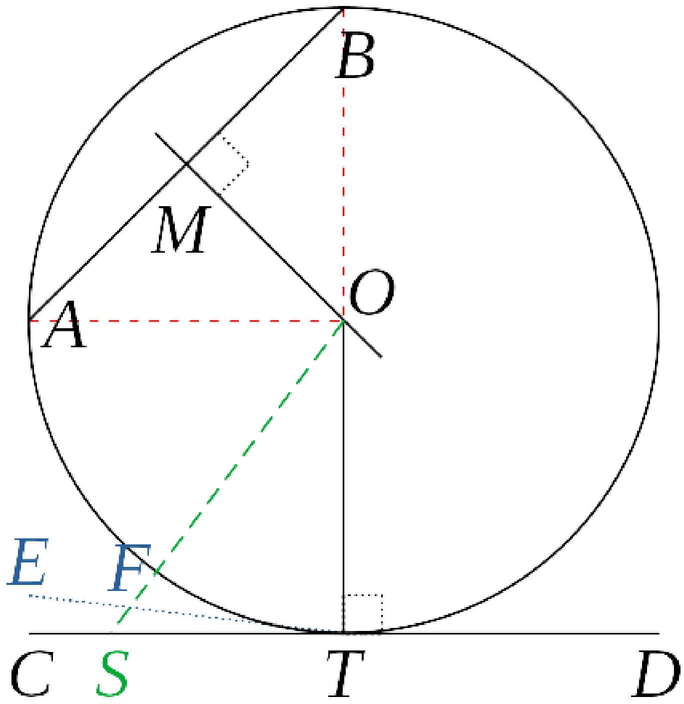
Mathematics 2260H – Geometry I: Euclidean Geometry
Trent University, Winter 2021

Solutions to Assignment #5
Circles and Right Angles
Due on Friday, 26 February.



Using nothing but what you have in Book I of the *Elements*, do each of the following questions. In both cases, we will call the centre of the given circle O.

1. Suppose A and B are different points on a circle. Show that the perpendicular bisector of the chord AB passes through the centre O of the circle. *[5]*

Solution. Draw the radii OA and OB, let M be the midpoint of AB, and draw the line OM as well. Since $|OA| = |OB|$ (as both are radii of the circle), $|AM| = |BM|$ (as M is the midpoint of AB), and $|OM| = |OM|$ (by one of those pesky common notions), it follows by the SSS congruence criterion (Proposition I-8) that $\triangle AMO \cong \triangle BMO$. It follows that $\angle AMO = \angle BMO$, and since $\angle AMO + \angle BMO = \angle AMB$ is straight angle, it follows by the definition of right angle that each of $\angle AMO$ and $\angle BMO$ is a right angle. Since this means that OM is perpendicular to AB and intersects AB at its midpoint M, it follows that OM is the perpendicular bisector of the chord AB. As the centre O of the circle is certainly on OM, we are done. ■

2. Suppose the line segment CD is tangent to the circle at T. Show that CD is perpendicular to the radius OT of the circle. *[6]*

Solution. We will first prove the converse to the given problem: if a line meets a circle at right angles to the radius at that point, then it is tangent to the circle. This done, we will show that there cannot be another tangent line at that point.

Suppose the line CD meets the radius OT at T and is perpendicular to OT. Let S be any point on the line CD other than T and consider $\triangle OST$. Since any two angles of a triangle sum to less than two right angles (Proposition I-17) and $\angle OTS$ is a right angle, $\angle OST < \angle OTS$. In any triangle the greater angle is subtended by the greater side (Proposition I-19), so $|OT| < |OS|$. Since OT is a radius of the circle, it follows that OS is not a radius of the circle, so S is not on the circle. Thus the line CD meets the circle only at T, making it a tangent line of the circle.

It remains to show that the tangent line to the circle at T in the paragraph above is the only tangent line the circle has at T. Suppose, by way of contradiction, that ET is a tangent line to the circle other than CD that passes through T. Without loss of generality, we may suppose that E is on the same side of CD as O and on the same side of OT as C. (If not, we may use D in place of C and/or extend ET past T to some point G and work with GT in place of ET.) Let S be a point on CD on the same side of T as C is such that $\angle SOT < \angle CTE$, and let F be the intersection of OS with ET.

Consider the internal angles of $\triangle OFT$. $\angle FOT = \angle SOT < \angle CTE = \angle STF$ by the choice of S (and hence of F) in the paragraph above. $\angle FTO = \angle STO - \angle STF = \angle CTO - \angle CTE$, which is less that a right angle because $\angle CTO$ is a right angle. Since F must be outside the circle (otherwise the tangent line ET would have to intersect the circle again once extended far enough), $|OF| > |OT|$, from which it follows by Proposition I-18 that $\angle CTO = \angle FTO > \angle OFT$. Since the sum of the internal angles of a triangle is equal to two right angles by Proposition I-32, we have that

$$\pi\, rad = \angle FOT + \angle FTO + \angle OFT < \angle CTE + (\angle CTO - \angle CTE) + \angle CTO = \pi\, rad\,,$$

which is a contradiction. Thus there cannot be another tangent line to the circle at T.

Since the line through T perpendicular to OT is tangent to the circle and it is the only line tangent line to the circle at T, it follows that if CD is tangent to the circle at T, then CD is perpendicular to the radius OT, as required. ■

NOTE. **1** is the Corollary given in the *Elements* to Proposition III-1. Similarly, **2** is essentially the Corollary given in the *Elements* to Proposition III-16. The proofs given above are a bit different since they are direct from the material in Book I, but use ideas similar to those used in the proofs of the propositions the corollaries follow from.