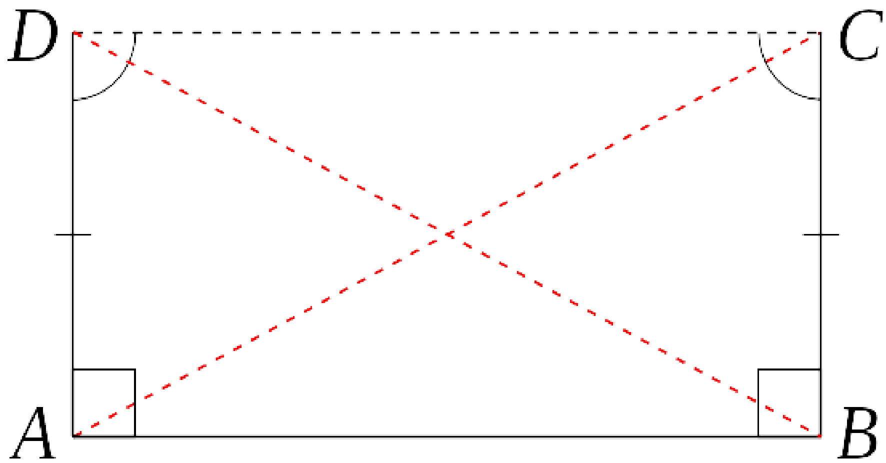
Mathematics 2260H – Geometry I: Euclidean Geometry
TRENT UNIVERSITY, Winter 2021

Solutions to Assignment #4
Saccheri quadrilaterals
Due on Friday, 12 February.



A *Saccheri quadrilateral* is a would-be rectangle, namely a quadrilateral that has two equal sides perpendicular to the base. In the diagram above the base is AB and we have $\angle DAB = \angle CBA = \dfrac{\pi}{2}$ *rad* and $|AD| = |BC|$.

1. Without using Postulate V or an equivalent, show that $\angle ADC = \angle BCD$. *[4]*

SOLUTION. Draw the diagonals AC and BD of the given Saccheri quadrilateral and consider the triangles $\triangle ABD$ and $\triangle BAC$. Since $|AD| = |BC|$, $|AB| = |BA|$, and $\angle DAB = \angle CBA$, $\triangle ABD \cong \triangle BAC$ by the SAS congruence criterion (Proposition I-4). It follows, in particular, that $|AC| = |BD|$.

Now consider the triangles $\triangle ADC$ and $\triangle BCD$. Since $|AD| = |BC|$, $|AC| = |BD|$, and $|DC| = |CD|$, it follows by the SSS congruence criterion (Proposition I-8), that $\triangle ADC \cong \triangle BCD$. It follows that $\angle ADC = \angle BCD$, as desired. ■

That's as much as can be done without applying Postulate V or an equivalent.

2. Using Postulate V or an equivalent, show that $\angle ADC$ and $\angle BCD$ are right angles and that $|AB| = |CD|$, making $ABCD$ a rectangle. *[6]*

SOLUTION. AB is a straight line falling across the straight lines AD and BC, making internal angles that add up to two right angles. (Well, they *are* two right angles ... :-) It follows by Proposition I-28 that $AD \parallel BC$. (Note that Proposition I-28 does not require Postulate V.) By Proposition I-33 it follows that $AB \parallel DC$ and $|AB| = |DC|$. (Proposition I-33 does depend on Postulate V by way of Proposition I-29.)

As AD is thus a straight line falling across the parallel straight lines AB and DC, so it makes internal angles on the same side that add up to right angles by (the non-Z-Theorem part of) Proposition I-29 (which depends on Postulate V). Since $\angle DAB$ is a right angle by hypothesis, it follows that $\angle ADC$ must also be a right angle. Similar reasoning starting from CB being a straight line falling across the parallel straight lines AB and DC shows that $\angle BCD$ is also a right angle.

Since $ABCD$ is a parallelogram whose internal angles are all right angles, it is a rectangle. ■

NOTE: Saccheri quadrilaterals are named after Giovanni Saccheri (1667-1733), a Jesuit priest and mathematician who attempted to show that Postulate V followed from the other Postulates by trying to show that denying Postulate V led to contradictions. Some of his ideas, and his use of these quadrilaterals in particular, were anticipated by the Persian poet and mathematician Omar Khayyam (1048-1131).