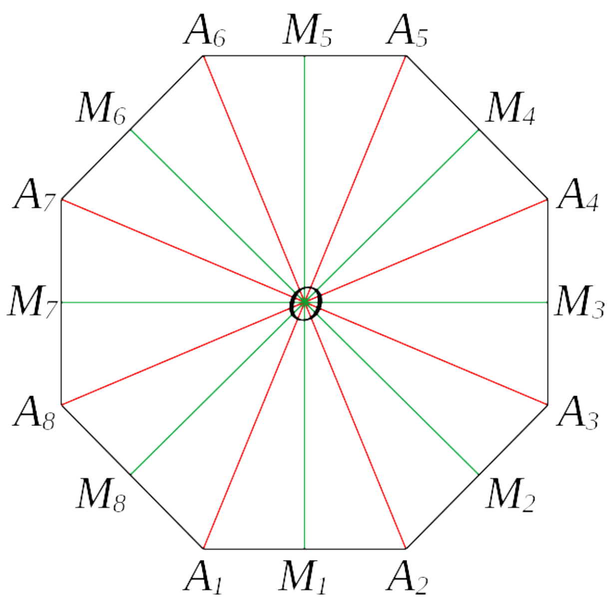
**Mathematics 2260H – Geometry I: Euclidean Geometry**
Trent University, Winter 2021

**Solutions to Assignment #3**
**A centre for a regular $n$-gon**
*Due on Friday, 5 February.*

Recall that a regular polygon is one with all sides of equal length and all internal angles equal. A polygon with $n$ sides is often referred to as an *$n$-gon*.* In what follows, suppose $A_1A_2 \ldots A_n$ is a regular $n$-gon in the Euclidean plane for some $n \geq 3$.

**1.** Let $\ell_1$, $\ell_2$, ... , and $\ell_n$ be the lines bisecting (*i.e.* cutting in half) the interior angles at $A_1$, $A_2$, ... , and $A_n$, respectively, of the regular $n$-gon $A_1A_2 \ldots A_n$. Show that $\ell_1$, $\ell_2$, ... , and $\ell_n$ are *concurrent*, that is meet at a common point $O$. *[4]*

Solution. Here is diagram of the case $n = 8$:




We will not only show that the angle bisectors are all concurrent at a point $O$, but that this point is equidistant from all the vertices of the polygon. The latter fact will help with questions **2** and **3**.

Since $A_1A_2 \ldots A_n$ is a regular $n$-gon, each internal angle must be less than a straight angle. It follows that the bisectors of the equal internal angles at $A_1$ and $A_2$, namely $\ell_1$ and $\ell_2$, make with $A_1A_2$ are equal to each other and each is less than half a straight angle, *i.e.* each is less than a right angle. Thus $A_1A_2$ is a line falling across the lines $\ell_1$ and $\ell_2$,

* For small $n$ we have common names: triangle, quadrilateral, pentagon, and so on. Note that in the Euclidean and hyperbolic planes an $n$-gon with positive area must have $n \geq 3$, but in the elliptic plane there are 2-gons ("biangles"?) with positive area.

with the internal angles on one side adding up to less than two right angles, so $\ell_1$ and $\ell_2$ must intersect on that side of $A_1A_2$ (the side the rest of the polygon is on) by Postulate V. Call this point of intersection $O$. Note that since $\angle OA_1A_2 = \angle OA_2A_1$, it follows by Proposition I-6 that $|OA_1| = |OA_2|$, *i.e* $\triangle OA_1A_2$ is isosceles.

We can apply the same reasoning to $A_2$, $A_3$, $\ell_2$, and $\ell_3$ to get that $\ell_2$ and $\ell_3$ intersect at some point $P$. We claim that $P = O$. Since $|A_1A_2| = |A_2A_3|$ (since the polygon is regular) and $\angle OA_1A_2 = \angle OA_2A_1 = \angle PA_2A_3 = \angle PA_3A_2$ (since each is half of an internal angle of the regular polygon, all of which are equal), it follows by the ASA congruence criterion (Proposition I-26) that $\triangle OA_1A_2 \cong \triangle PA_3A_2$. This means that $|OA_2| = |PA_2|$. Since $O$ and $P$ are both on the angle bisector $\ell_2$ on the same side of $A_2$ and an equal distance from $A_2$, $O$ and $P$ must be the same point. Since $\triangle OA_1A_2 \cong \triangle OA_3A_2$ and the former triangle is isosceles, so is the other. In particular, $|OA_1| = |OA_2| = |OA_3|$

We can apply the reasoning in the paragraph above to successively show that $\ell_3$ and $\ell_4$ intersect at $O$, $\ell_4$ and $\ell_5$ intersect at $O$, and so on. Thus all the angle bisectors $\ell_1$, $\ell_2$, ..., and $\ell_n$ are concurrent at $O$. Moreover, all the vertices are equidistant from $O$, *i.e.* $|OA_1| = |OA_2| = \cdots = |OA_n|$. ■

**2.** Let $m_1$, $m_2$, ..., and $m_n$ be the perpendicular bisectors (*i.e.* lines cutting in half at a right angle) of the sides $A_1A_2$, $A_2A_3$, ..., and $A_nA_1$, respectively, of the regular $n$-gon $A_1A_2 \dots A_n$. Show that $m_1$, $m_2$, ..., and $m_n$ are also concurrent at the point $O$ in question **1**. *[4]*

SOLUTION. Let $M_1$, $M_2$, ..., $M_n$ be the midpoints of the sides $A_1A_2$, $A_2A_3$, ..., $A_nA_1$, respectively, so each perpendicular bisector $m_i$ passes through the midpoint $M_i$ of the corresponding side. We will show that the perpendicular bisectors are all concurrent at $O$, and also that $O$ is equidistant from all the midpoints $M_i$. The latter fact will be used in the solution to **3**.

The line $A_1M_1$ falls across the lines $\ell_1$ and $m_1$, with the internal angles being less than a right angle (at $A_1$) and a right angle (at $M_1$) on the side of $A_1A_2$ that $O$ is on. Since the sum of these internal angles is then less than two right angles, it follows by Postulate V that $\ell_1$ and $m_1$ meet at some point $P$ on the same side of $A_1A_2$ as $O$. Similar reasoning will show that $m_1$ and $\ell_2$ will meet at some point $Q$ on the same side of $A_1A_2$ as $O$. Since $|A_1M_1| = |A_2M_1|$ (because $m_1$ bisects $A_1A_2$), $\angle PA_1M_1 = \angle QA_2M_1$ (because each is half of an equal internal angle of the polygon), and $\angle PM_1A_1 = \angle QM_1A_2$ (because each is a right angle as $m_1$ is perpendicular to $A_1A_2$), it follows by the ASA congruence criterion (Proposition I-26), that $\triangle PA_1M_1 \cong QA_2M_1$. This means that $|PM_1| = |QM_1|$. Since $P$ and $Q$ are both points on $m_1$, on the same side of $A_1A_2$ as $O$, it follows that $P = Q$. Further, since $\ell_1$ passes through $P = Q$ and $\ell_2$ passes through $Q = P$, and $\ell_1$ and $\ell_2$ intersect at $O$, it follows that $P = Q = O$. Thus $m_1$ passes through $O$.

One can repeat the above reasoning for perpendicular bisector $m_i$ of a side of the regular polygon, showing that they are all concurrent on $O$. Note that the triangles $\triangle OA_iM_i$ are all congruent to each other by the ASA congruence criterion (checking this is left to you!), so all the line segments $OM_i$ are equal in length, *i.e.* $O$ is equidistant from the midpoints of the sides of the given regular polygon. ■

**3.** Besides the regular polygon $A_1A_2 \dots A_n$, what else is the point $O$ a centre of? *[2]*

Solution. $O$ is the centre of a circle passing through the vertices of the regular $n$-gon $A_1A_2 \dots A_n$ because, as was shown in the solution to **1** above, $O$ is the same distance from each vertex $A_i$. $O$ is also the centre of a circle passing through the midpoints of the sides of the regular $n$-gon $A_1A_2 \dots A_n$ because, as was shown in the solution to **2** above, $O$ is an equal distance from each midpoint $M_i$. ■