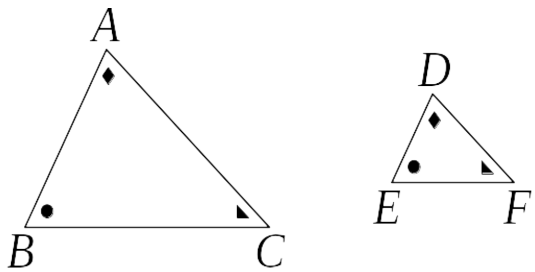
Mathematics 2260H – Geometry I: Euclidean Geometry
Trent University, Winter 2021

Solutions to Assignment #2 – Similarity
Due on Friday, 29 January.

Two triangles are said to be *similar* if they have the same proportions, though not necessarily the same size. More formally, $\triangle ABC$ is similar to $\triangle DEF$, often written as $\triangle ABC \sim \triangle DEF$, if corresponding angles of the triangles are equal, *i.e.* $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$.



Similarity is a very useful tool that we will develop a little earlier than Euclid does in the *Elements* by using trigonometry. You may – and definitely should! – assume thoughout this assignment that the sum of the interior angles of a triangle is equal to two right angles. (Or a straight angle, or π radians, or $180°$, or 200 gradians, or ... :-)

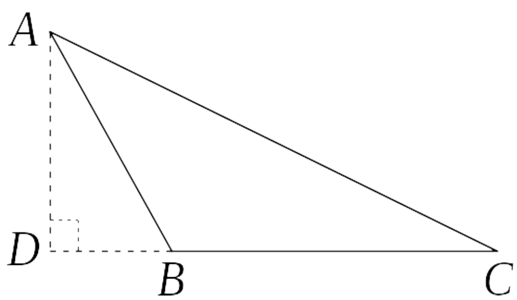
1. Prove the sine law for triangles, *i.e.* given any $\triangle ABC$,

$$\frac{\sin(\angle BAC)}{|BC|} = \frac{\sin(\angle ABC)}{|AC|} = \frac{\sin(\angle ACB)}{|AB|},$$

and the cosine law for triangles, *i.e.* given any $\triangle ABC$,

$$|BC|^2 = |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle BAC) . \qquad [3]$$

Solution. We will go back to high school trigonometry to prove the Law of Sines. Suppose we are given $\triangle ABC$. Draw a line perpendicular to BC from A, meeting (an extension of) BC (if necessary) at D (Proposition I-12!), as in the diagram below.



Note that $\triangle ADB$ and $\triangle ADC$ are right triangles with $\angle ADC = \triangle ADB$ as the right angle(s). Note further that if $\angle ABC$ is obtuse, as in the given diagram, then, measuring angles in radians, $\angle ABD = \pi - \angle ABC$ and so $\sin(\angle ABC) = \sin(\pi - \angle ABC) = \sin(\angle ABD)$. If $\angle ABC$ were to be acute instead, then $\angle ABD = \angle ABC$.

It follows that

$$\sin(\angle ACB) = \sin(\angle ACD) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{|AD|}{|AC|}$$
$$\sin(\angle ABC) = \sin(\angle ABD) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{|AD|}{|AB|}$$

Thus $|AC|\sin(\angle ACB) = |AD| = |AB|\sin(\angle ABC)$. Dividing on both ends of the equation by $|AC| \cdot |AB|$ now gives us $\dfrac{\sin(\angle ABC)}{|AC|} = \dfrac{\sin(\angle ACB)}{|AB|}$. A similar argument will show that $\dfrac{\sin(\angle BAC)}{|BC|} = \dfrac{\sin(\angle ABC)}{|AC|}$, completing the proof of the Law of Sines.

I really can't improve on the presentation of the standard trigonometric proof from the Wikipedia article about the Law of Cosines:

Using trigonometry [edit]

Dropping the perpendicular onto the side c through point C, an altitude of the triangle, shows (see Fig. 5)

$$c = a\cos\beta + b\cos\alpha.$$

(This is still true if α or β is obtuse, in which case the perpendicular falls outside the triangle.) Multiplying through by c yields

$$c^2 = ac\cos\beta + bc\cos\alpha.$$

Considering the two other altitudes of the triangle yields

$$a^2 = ac\cos\beta + ab\cos\gamma,$$
$$b^2 = bc\cos\alpha + ab\cos\gamma.$$

Adding the latter two equations gives

$$a^2 + b^2 = ac\cos\beta + bc\cos\alpha + 2ab\cos\gamma.$$

Subtracting the first equation from the last one results in

$$a^2 + b^2 - c^2 = ac\cos\beta + bc\cos\alpha + 2ab\cos\gamma - (ac\cos\beta + bc\cos\alpha)$$

which simplifies to

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

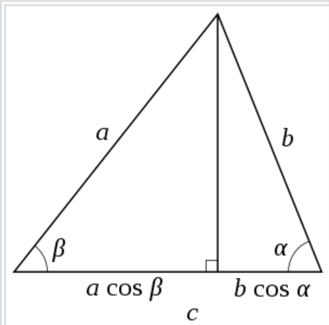


Fig. 5 – An acute triangle with perpendicular

This proof uses trigonometry in that it treats the cosines of the various angles as quantities in their own right. It uses the fact that the cosine of an angle expresses the relation between the two sides enclosing that angle in *any* right triangle. Other proofs (below) are more geometric in that they treat an expression such as $a\cos\gamma$ merely as a label for the length of a certain line segment.

$\square$

2. Show that if $\triangle ABC \sim \triangle DEF$, then $\dfrac{|AB|}{|DE|} = \dfrac{|BC|}{|EF|} = \dfrac{|AC|}{|DF|}$. *[2]*

Solution. Suppose $\triangle ABC \sim \triangle DEF$. By definition, this means that $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$. By the Law of Sines we have

$$\frac{\sin(\angle BAC)}{|BC|} = \frac{\sin(\angle ABC)}{|AC|} = \frac{\sin(\angle ACB)}{|AB|}$$
$$\text{and } \frac{\sin(\angle EDF)}{|EF|} = \frac{\sin(\angle DEF)}{|DF|} = \frac{\sin(\angle DFE)}{|DE|}.$$

Inverting the former equation and multiplying it into the latter, we get

$$\frac{\sin(\angle EDF)}{|EF|} \cdot \frac{|BC|}{\sin(\angle BAC)} = \frac{\sin(\angle DEF)}{|DF|} \cdot \frac{|AC|}{\sin(\angle ABC)} = \frac{\sin(\angle DFE)}{|DE|} \cdot \frac{|AB|}{\sin(\angle ACB)} .$$

Because the corresponding angles are equal, they cancel out, leaving us with

$$\frac{|BC|}{|EF|} = \frac{|AC|}{|DF|} = \frac{|AB|}{|DE|} ,$$

as required. $\square$

Note that it follows from **2** that if two triangles are similar and have one pair of corresponding sides equal, then the triangles are congruent. (Why?) [*Answer*: By one of the Angle-Side-Angle or Angle-Angle-Side congruence criteria, also known as Proposition I-26 between them.]

3. Show that if $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$, then $\triangle ABC \sim \triangle DEF$. *[2]*

Solution. Let $r = \frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$. By the Law of Cosines, we have

$$\begin{aligned} |BC|^2 &= |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle BAC) \\ \text{and } |EF|^2 &= |DE|^2 + |DF|^2 - 2 \cdot |DE| \cdot |DF| \cdot \cos(\angle EDF) . \end{aligned}$$

Multiplying the latter equation by r^2 yields:

$$\begin{aligned} & r^2|EF|^2 = r^2|DE|^2 + r^2|DF|^2 - 2r^2 \cdot |DE| \cdot |DF| \cdot \cos(\angle EDF) \\ \Longrightarrow & \left(\frac{|BC|}{|EF|}\right)^2 |EF|^2 = \left(\frac{|AB|}{|DE|}\right)^2 |DE|^2 + \left(\frac{|AC|}{|DF|}\right)^2 |DF|^2 \\ & \qquad - 2 \cdot \frac{|AB|}{|DE|} \cdot \frac{|AC|}{|DF|} \cdot |DE| \cdot |DF| \cdot \cos(\angle EDF) \\ \Longrightarrow & |BC|^2 = |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle EDF) \end{aligned}$$

It follows that

$$\cos(\angle BAC) = \frac{|BC|^2 - |AB|^2 - |AC|^2}{-2 \cdot |AB| \cdot |AC|} = \cos(\angle EDF) .$$

Since the interior angles in a triangle are each between 0 and two right angles and the cosine function is 1–1 on this domain, it follows that $\angle BAC = \angle EDF$. Similar arguments can be used to show that $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$, so the triangles are similar, *i.e.* $\triangle ABC \sim \triangle DEF$, by definition. $\square$

4. Show that if $\angle BAC = \angle EDF$ and $\dfrac{|AB|}{|DE|} = \dfrac{|AC|}{|DF|}$, then $\triangle ABC \sim \triangle DEF$. *[2]*

SOLUTION. Let $r = \dfrac{|AB|}{|DE|} = \dfrac{|AC|}{|DF|}$. By the Law of Cosines, we have

$$\begin{aligned}|BC|^2 &= |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle BAC)\\ \text{and } |EF|^2 &= |DE|^2 + |DF|^2 - 2 \cdot |DE| \cdot |DF| \cdot \cos(\angle EDF)\ .\end{aligned}$$

Multiplying the latter equation by r^2 yields and using $\angle BAC = \angle EDF$:

$$\begin{aligned}r^2|EF|^2 &= r^2|DE|^2 + r^2|DF|^2 - 2r^2 \cdot |DE| \cdot |DF| \cdot \cos(\angle EDF)\\ &= \left(\frac{|AB|}{|DE|}\right)^2 |DE|^2 + \left(\frac{|AC|}{|DF|}\right)^2 |DF|^2\\ &\qquad - 2 \cdot \frac{|AB|}{|DE|} \cdot \frac{|AC|}{|DF|} \cdot |DE| \cdot |DF| \cdot \cos(\angle BAC)\\ &= |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle BAC)\\ &= |BC|^2\end{aligned}$$

It follows that $r^2 = \dfrac{|BC|^2}{|EF|^2}$, and so $r = \dfrac{|BC|}{|EF|} = \dfrac{|AB|}{|DE|} = \dfrac{|AC|}{|DF|}$. It now follows from **3** that the triangles are similar, *i.e.* $\triangle ABC \sim \triangle DEF$. □

3 is the Side-Side-Side similarity criterion and **4** is the Side-Angle-Side similarity criterion for triangles. **5** is a congruence problem in which similarity may be useful:

5. Determine whether the Angle-Angle-Side congruence criterion for triangles always works. That is, given that $\angle CAB = \angle FDE$, $\angle ABC = \angle DEF$, and $|BC| = |EF|$, does it necessarily follow that $\triangle ABC \cong \triangle DEF$? *[1]*

SOLUTION. As noted above, Proposition I-26 includes both the Side-Angle-Side and Angle-Angle-Side congruence criteria, and has a proof in the *Elements*.

One could also prove the Angle-Angle-Side congruence criterion directly by observing that since the sum of the interior angles of any triangle is equal to right angles, it follows from the hypotheses that $\angle BAC = \angle EDF$, so $\triangle ABC \sim \triangle DEF$. Since the triangles are similar and $|BC| = |EF|$, we also have $\dfrac{|AB|}{|DE|} = \dfrac{|AC|}{|DF|} = \dfrac{|BC|}{|EF|} = 1$ by **2**, so $|AB| = |DE|$ and $|AC| = |DF|$. Since the triangles have corresponding angles and sides equal, they are congruent by definition, *i.e.* $\triangle ABC \cong \triangle DEF$. ■