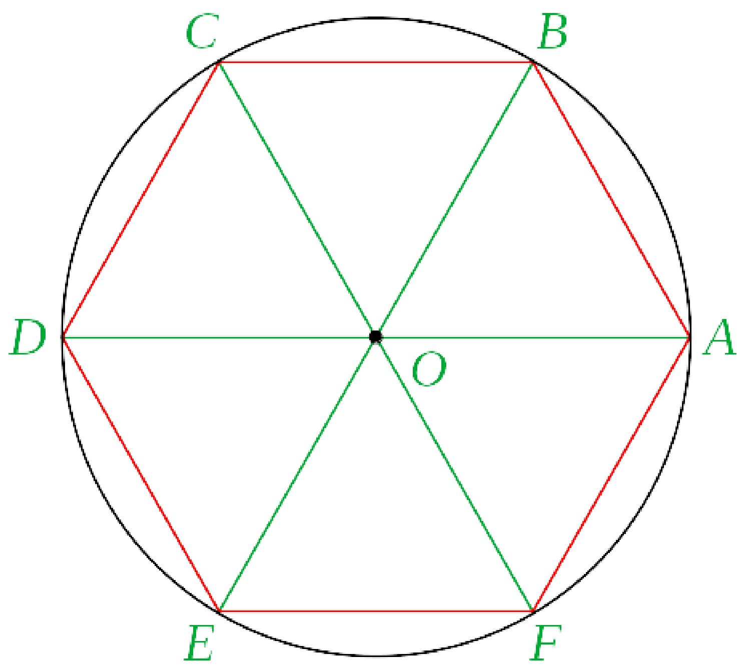
Mathematics 2260H – Geometry I: Euclidean Geometry
Trent University, Winter 2021

Solution to Assignment #1 – Hex-agony?
Due on Friday, 22 January.



1. Suppose one is given a circle and its centre in the Euclidean plane. Using the more complete version of Euclid's postulates given in the handout *Euclids Postulates Extended*, show how to construct a regular hexagon* inscribed† in the given circle. Make sure to explain why your construction works as part of the construction or separately. *[10]*

Solution. Finding a construction that works is relatively easy, and showing that it works is relatively hard. Here is one of a number of possible constructions. Before we begin, let us denote the centre of the circle by O.

1. Pick a point A on the circle and draw the radius OA using Postulate I.
2. Use Proposition I-1 to construct an equilateral triangle $\triangle OAB$ on OA. Since the triangle is equilateral, $|OB| = |OA| = |AB|$, so B is also on the circle by the definition of a circle and OB is also a radius.
3. Use Proposition I-1 to construct an equilateral triangle $\triangle OBC$ on OB. Since the triangle is equilateral, $|OC| = |OB| = |BC|$, so C is also on the circle by the definition of a circle and OC is also a radius.

---

* A *regular polygon* is a polygon in which all the sides are the same length and all the interior angles are equal to each other.

† A polygon is *inscribed* in a circle if all of its vertices are on the circle.

4. Use Proposition I-1 to construct an equilateral triangle $\triangle OCD$ on OC. Since the triangle is equilateral, $|OD| = |OC| = |CD|$, so D is also on the circle by the definition of a circle and OD is also a radius.
5. Use Proposition I-1 to construct an equilateral triangle $\triangle ODE$ on OD. Since the triangle is equilateral, $|OE| = |OD| = |DE|$, so E is also on the circle by the definition of a circle and OE is also a radius.
6. Use Proposition I-1 to construct an equilateral triangle $\triangle OEF$ on OE. Since the triangle is equilateral, $|OF| = |OE| = |EF|$, so F is also on the circle by the definition of a circle and OF is also a radius.
7. Draw the line segment FA using Postulate I.

We claim that $ABCDEF$ is a regular hexagon inscribed the given circle.

The hexagon $ABCDEF$ is inscribed in the given circle because all six vertices are on the circle, as noted in the course of the construction above. It remains to show that $ABCDEF$ is a regular hexagon.

Note that the five triangles $\triangle OAB$, $\triangle OBC$, $\triangle OCD$, $\triangle ODE$, and $\triangle OEF$, are all equilateral triangles with the same side length, equal to the radius of the circle. It follows by the Side-Side-Side congruence criterion, *i.e.* Proposition I-8, that all five triangles are congruent to each other and hence all have the same interior angles too.

So far as the sides are concerned, we are most of the way there because $|AB| = |BC| = |CD| = |DE| = |EF|$, by the construction, since each is equal to a radius of the given circle. We still need to show that FA has the same length.

So far as the interior angles are concerned, we are also most of the way there since each of $\angle ABC = \angle ABO + \angle OBC$, $\angle BCD = \angle BCO + \angle OCD$, $\angle CDE = \angle CDO + \angle ODE$, and $\angle DEF = \angle DEO + \angle OEF$ is equal to the sum of two (equal) interior angles of the congruent equilateral triangles, and hence are all equal to each other. We still need to show that $\angle EFA$ and $\angle FAB$ are equal to the other four interior angles.

The key to completing the job will be to show that $\triangle OFA$ is congruent to each of the five equilateral triangles we constructed. It will then immediately follow that AF has the same length as the other five sides of the hexagon, and that $\angle EFA = \angle EFO + \angle OFA$ and $\angle FAB = \angle FAO + \angle OAB$ are equal to the other four interior angles.

However, showing that $\triangle OFA$ is an equilateral triangle congruent to the five we constructed requires using Postulate V or a suitable equivalent. The equivalent we will use here is the assertion that the sum of the interior angles of a triangle is equal to two right angles, *i.e.* a straight angle. (This is a commonplace fact in standard trigonometry, so it should at least be familiar, even if we haven't gotten to it in the *Elements* yet.) Applied to an equilateral triangle, in which all three interior angles are equal, this means that the interior angles at each vertex is one third of a straight angle (or two thirds of a right angle). It follows that $\angle COF = \angle COD + \angle DOE + \angle EOF$ and $\angle DOA = \angle AOC + \angle COB + \angle BOA$ are both straight angles, so COE and DOA are straight lines. It follows by the Opposite Angles Theorem (Proposition I-15) that $\angle AOF = \angle COD$. Since we have $|OC| = |OF|$ and $|OD| = |OA|$, it follows by the Side-Angle-Side congruence criterion (Proposition I-4) that $\triangle OFA$ is congruent to $\triangle OCD$ and hence also the other four equilateral triangles.

This completes the proof. ■

Bonus. Construct a regular hexagon without using anything equivalent to Postulate V and without looking it up. *[+1% on your final mark]*