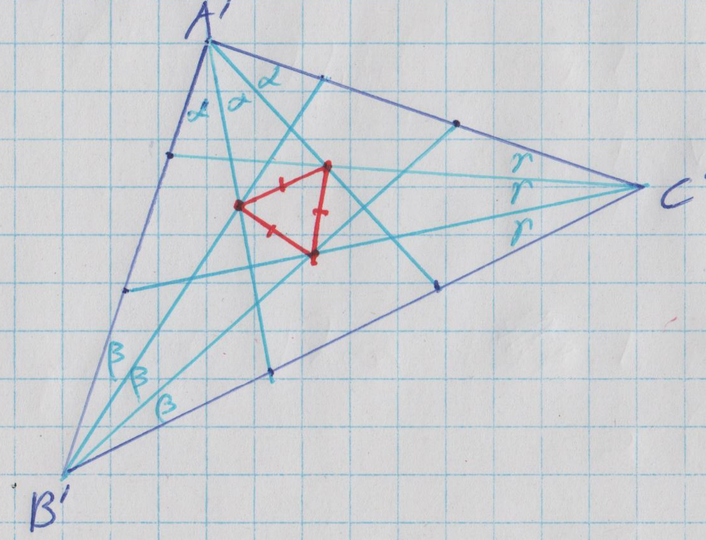
MATH 2260H

Morley's Trisector Theorem

2021-03-31



Recall that the angle bisectors of a triangle meet in a single point.

Thm: (Morley, 1899) In any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.

proof: (John Conway, 1995) Suppose that $\Delta A'B'C'$ has angles 3α, 3β, and 3γ at A', B' & C', respectively. Observe that since $3\alpha + 3\beta + 3\gamma = \pi$ rad., we have $\alpha+\beta+\gamma = \frac{\pi}{3}$. We can construct triangles with internal angles as follows, to any scale we desire, since the triples of angles add up to π radians in each case:

(0) $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$ An equilateral triangle scaled to have sides of length 1.

(1) $\alpha, \beta+\frac{\pi}{3}, \gamma+\frac{\pi}{3}$ Scaled so that the side opposite α has length 1.

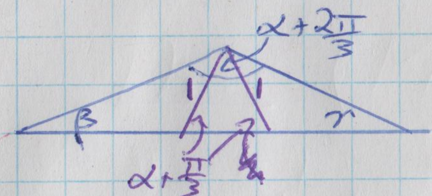
(2) $\alpha+\frac{\pi}{3}, \beta, \gamma+\frac{\pi}{3}$ —— " —— β —— " ——

(3) $\alpha+\frac{\pi}{3}, \beta+\frac{\pi}{3}, \gamma$ —— " —— γ —— " ——.

(4) $\alpha+\frac{2\pi}{3}, \beta, \gamma$

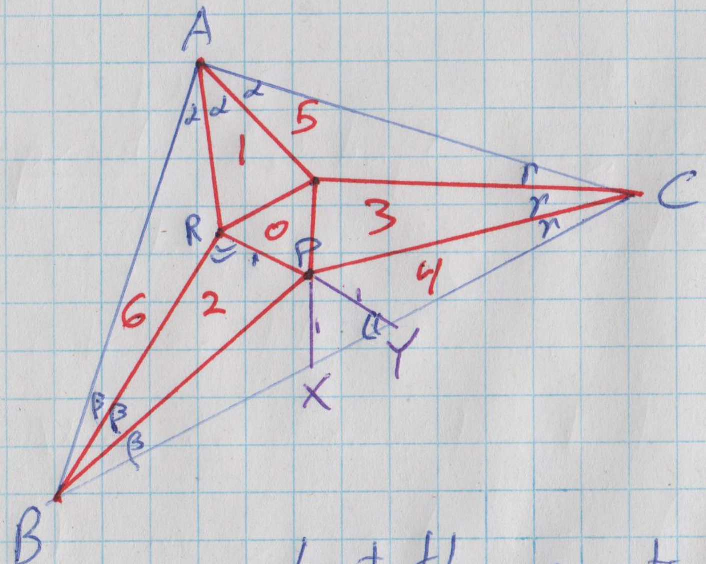
(5) $\alpha, \beta+\frac{2\pi}{3}, \gamma$

(6) $\alpha, \beta, \gamma+\frac{2\pi}{3}$



(4) is scaled so that ~~the~~ each line from the vertex with angle $\alpha+\frac{2\pi}{3}$ to the opposite side meeting this side at an angle of $\alpha+\frac{\pi}{3}$ has length 1

& similarly for (5) & (6).



Assemble these triangles as in the diagram. No problem with attaching triangles 1, 2, and 3 to triangle 0.

Why do triangle 4, 5, & 6 fit flush in here? Let's do this for 4.

Let the points be labelled as in the diagram. We have $|PR| = 1 = |PY|$, $\angle PBR = \beta = \angle PBY$, $\angle BRP = \alpha+\frac{\pi}{3} = \angle BYP$. So by AAS congruence, $\Delta BPR \cong \Delta BPY$, so $|BP| = |BP|$ in both (2) & (4).

Similarly, we can show that $|BP| = |CP|$ in both (4) & (3). Thus triangle 4 fits in. Similar arguments show that (5) & (6) fit into the picture too.

By the construction, ΔABC has the same internal angles at each vertex as $\Delta A'B'C'$ that we started with. Thus $\Delta ABC \sim \Delta A'B'C'$. It follows that the (scaled) versions of the triangle formed by the intersections of adjacent trisectors in each triangle, must be equilateral. //