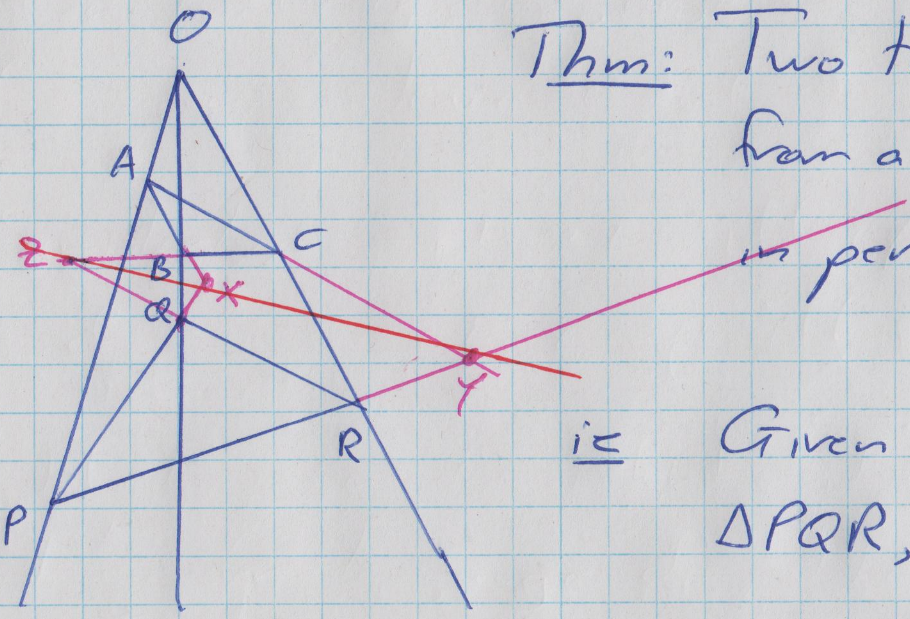
MATH 2260H <u>Desargues' Theorem</u>



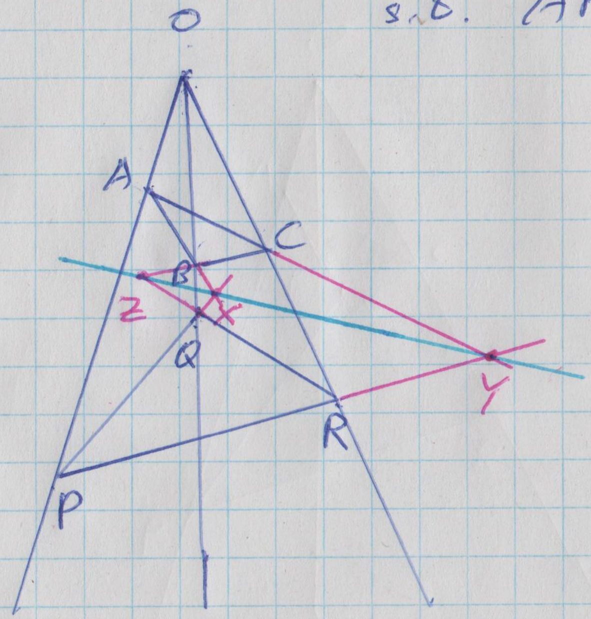
<u>Thm:</u> Two triangles are in perspective from a point <u>iff</u> they are in perspective from a line ℓ.

<u>ie</u> Given two triangles ΔABC and ΔPQR, AP, BQ, and CR are concurrent <u>iff</u> $X = AB \cap PQ$, $Y = AC \cap PR$, and $Z = BC \cap QR$ are collinear.

[This assumes there is no parallelism in the picture; the theorem still works if there is - just some (or all) of the points O, X, Y, & Z are "at infinity".]

We'll use Menelaus' Thm to prove this result. (It can also be proven from Pappus Thm.)

proof: $\boxed{\Rightarrow}$ Suppose that we are given ΔABC & ΔPQR s.t. AP, BQ, & CR are concurrent in a point O.



Let $X = AB \cap PQ$, $Y = AC \cap PR$, and $Z = BC \cap QR$. [We need to show that X, Y, Z are collinear.]

It would be sufficient, by Menelaus' Theorem, to show that either

$$\frac{AX}{XB} \cdot \frac{BZ}{ZC} \cdot \frac{CY}{YA} = -1 \quad \text{or} \quad \frac{PX}{XQ} \cdot \frac{QZ}{ZR} \cdot \frac{RY}{YP} = -1.$$

Lots of triangles:

ΔOAC : P, R, Y collinear $\Rightarrow \frac{OP}{PA} \cdot \frac{AY}{YC} \cdot \frac{CR}{RO} = -1$

ΔOAB : P, Q, X collinear $\Rightarrow \frac{OP}{PA} \cdot \frac{AX}{XB} \cdot \frac{BQ}{QO} = -1$

ΔOBC : Q, R, Z collinear $\Rightarrow \frac{OQ}{QB} \cdot \frac{BZ}{ZC} \cdot \frac{CR}{RO} = -1$

By the other direction of Menelaus' Thm.

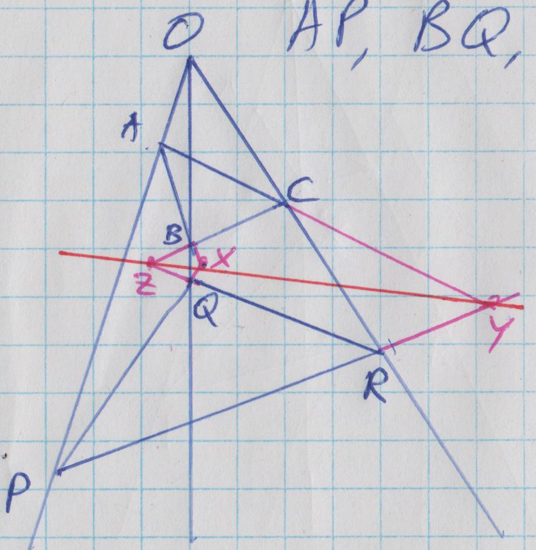
Then

$$\frac{AX}{XB} \cdot \frac{BZ}{ZC} \cdot \frac{CY}{YA} = \frac{\left(\cancel{\frac{OP}{PA}} \cdot \frac{AX}{XB} \cdot \cancel{\frac{BQ}{QO}}\right)\left(\cancel{\frac{OQ}{QB}} \cdot \frac{BZ}{ZC} \cdot \cancel{\frac{CR}{RO}}\right)}{\cancel{\frac{OP}{PA}} \cdot \frac{AY}{YC} \cdot \cancel{\frac{CR}{RO}}} = \frac{(-1)(-1)}{(-1)} = -1$$

& so $X, Y,$ & Z are collinear by Menelaus' Thm.

$\boxed{\Leftarrow}$

Suppose that $X = AB \cap PQ$, $Y = AC \cap PR$, and $Z = BC \cap QR$ are collinear. [We need to show that AP, BQ, and CR are concurrent.]



It is sufficient, by Menelaus' Thm, to show that (considering ΔXAP) and $O = PA \cap QB$,

I've lost the thread, so we'll finish this off next time...