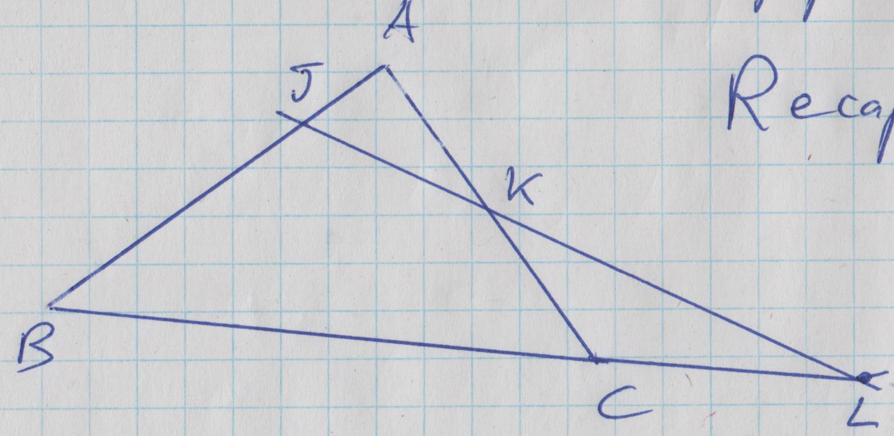
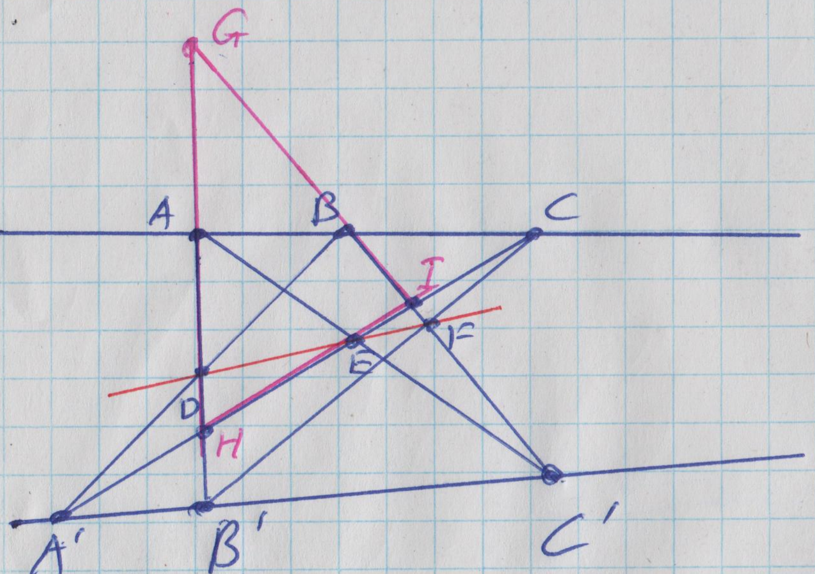
MATH 2260H Pappus' Theorem



Recap: Menelaus' Thm.

If J, K, & L are on sides AB, AC, & BC resp., then J, K, & L are collinear iff $\frac{AJ}{JB} \cdot \frac{BL}{LC} \cdot \frac{CK}{KA} = -1$.



Pappus Thm: Suppose A, B, & C are collinear and that A', B', C' are collinear (on another line).

Let $D = AB' \cap A'B$,
$E = AC' \cap A'C$,
& $F = BC' \cap B'C$.

Assuming that AB' & A'B etc are not parallel.

Then D, E, & F are collinear.

proof: Let H be $AB' \cap A'C$ and let I be $BC' \cap A'C$. Also, let G be $AB' \cap BC'$. Consider the ΔGHI.

1) Since each of A', D, B are on a side (or an extension of one) of ΔGHI and A', D, B are collinear,

$$\frac{GD}{DH} \cdot \frac{HA'}{A'I} \cdot \frac{IB}{BG} = -1 \quad \text{by Menelaus' Thm.}$$

2) — " — A, E, C' — " —
— " — A, E, C' — " — ,

$$\frac{GA}{AH} \cdot \frac{HE}{EI} \cdot \frac{IC'}{C'G} = -1 \qquad \text{— " —}$$

3) — " — C, F, B' — " —
— " — C, F, B' — " —

$$\frac{AB'}{B'H} \cdot \frac{HC}{CI} \cdot \frac{IF}{FG} = -1 \qquad \text{— " —}$$

4) — " — A, B, C — " —
— " — A, B, C — " —

$$\frac{GA}{AH} \cdot \frac{HC}{CI} \cdot \frac{IB}{BG} = -1 \qquad \text{— " —}$$

5) ——— " ——— A', B', C' ——— " ———

——— " — A', B', C' ——— " ———

$$\frac{GB'}{B'H}\cdot\frac{HA'}{A'I}\cdot\frac{IC'}{C'G} = -1.$$

Then $\left(\frac{GD}{DH}\cdot\cancel{\frac{HA'}{A'I}}\cdot\cancel{\frac{IB}{BG}}\right)\cdot\left(\cancel{\frac{GA}{AH}}\cdot\frac{HE}{EI}\cdot\cancel{\frac{IC'}{C'G}}\right)\cdot\left(\cancel{\frac{GB'}{B'H}}\cdot\cancel{\frac{HC}{CI}}\cdot\frac{IF}{FG}\right)$

$(-1)^5 =$ $\quad\cdot\left(\cancel{\frac{GA}{AH}}\cdot\cancel{\frac{HC}{CI}}\cdot\cancel{\frac{IB}{BG}}\right)\cdot\left(\cancel{\frac{GB'}{B'H}}\cdot\cancel{\frac{HA'}{A'I}}\cdot\cancel{\frac{IC'}{C'G}}\right)$

$\|$

-1 $\quad\cdot\left(\cancel{\frac{AH}{GA}}\cdot\cancel{\frac{CI}{HC}}\cdot\cancel{\frac{BG}{IB}}\right)\cdot\left(\cancel{\frac{B'H}{GB'}}\cdot\cancel{\frac{A'I}{HA'}}\cdot\cancel{\frac{C'G}{IC'}}\right)$

$$= \boxed{\frac{GD}{DH}\cdot\frac{HE}{EI}\cdot\frac{IF}{FG} = -1}$$

$\therefore$ D, E, F are collinear by Menelaus' Thm.

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