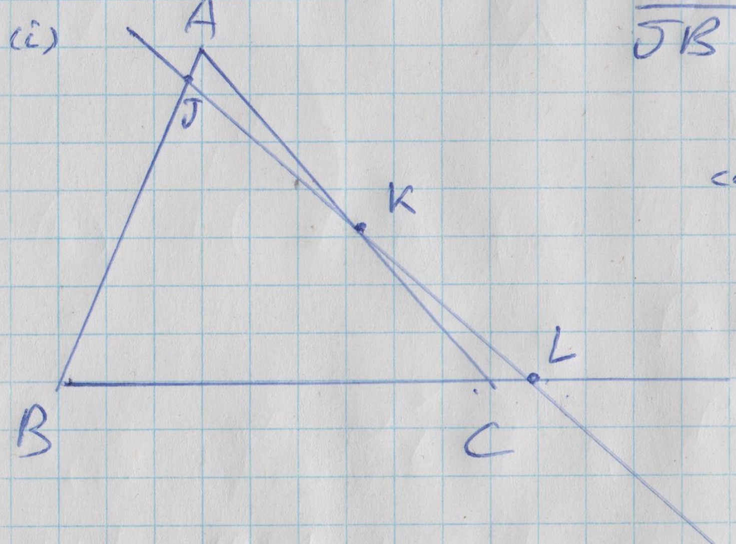
MATH 2260H Menelaus' Theorem

Recall: $\frac{AB}{BC} = \begin{cases} \frac{|AB|}{|BC|} & \text{if } B \text{ is between } A \,\&\, C \\ -\frac{|AB|}{|BC|} & \text{otherwise} \end{cases}$

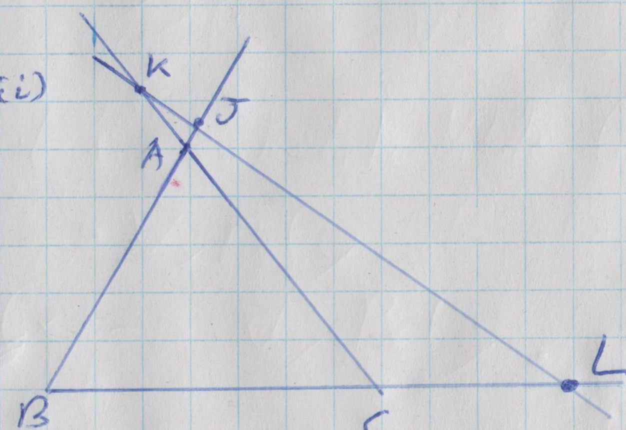
Menelaus' Thm.: Suppose J, K, & L are points on (extensions of) sides AB, AC, & BC of $\triangle ABC$, resp. Then J, K, & L are collinear if and only if

$$\frac{AJ}{JB} \cdot \frac{BL}{LC} \cdot \frac{CK}{KA} = -1.$$

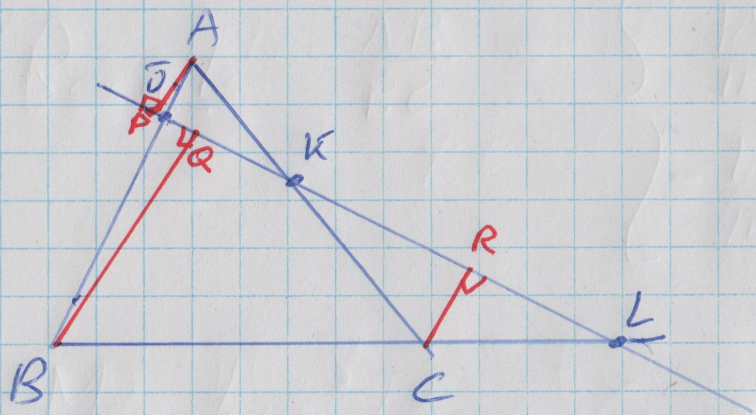
case(i)



case(ii)



proof: $\boxed{\Rightarrow}$ case (i) Suppose J, K, L are collinear

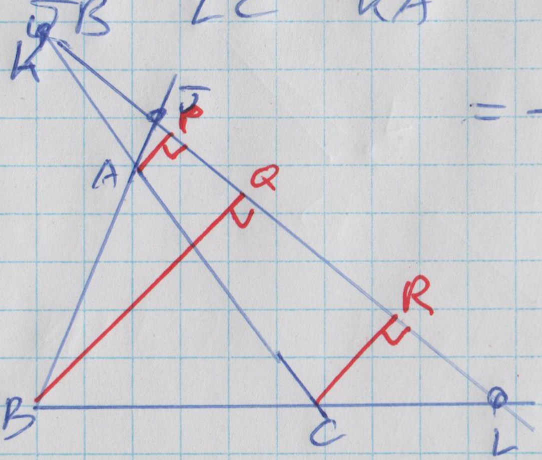


Draw perpendiculars AP, BQ, & CR from $A, B,$ & C to the line JK. Since they are all perpendicular to JK, they are parallel to each other. Observe that $\angle AJP = \angle BJQ$ (Opposite Angle Thm.), and $\angle JAP = \angle JAB = \angle ABQ = \angle JBQ$ (by the Z-Thm.), so $\Delta AJP \sim \Delta BJQ$, so (in turn) $\frac{|AJ|}{|JB|} = \frac{|AP|}{|BQ|}$. Similarly, we have $\Delta APK \sim \Delta CRK$, so $\frac{|CK|}{|KA|} = \frac{|CR|}{|AP|}$, and $\Delta BQL \sim \Delta CRL$, so $\frac{|BL|}{|CL|} = \frac{|BQ|}{|CR|}$. Then

$$\frac{AJ}{JB} \cdot \frac{BL}{LC} \cdot \frac{CK}{KA} = \left(+\frac{|AJ|}{|JB|}\right) \cdot \left(-\frac{|BL|}{|LC|}\right) \cdot \left(+\frac{|CK|}{|KA|}\right)$$

$$= -\frac{\cancel{|AP|}}{\cancel{|BQ|}} \cdot \frac{\cancel{|BQ|}}{\cancel{|CR|}} \cdot \frac{\cancel{|CR|}}{\cancel{|AP|}} = -1.$$

case (ii)



We get similarity as above, and so

$$\frac{AJ}{JB} \cdot \frac{BL}{LC} \cdot \frac{CK}{KA} = \left(-\frac{|AJ|}{|JB|}\right) \cdot \left(-\frac{|BL|}{|LC|}\right) \cdot \left(-\frac{|CK|}{|KA|}\right)$$

$$= (-1)^3 \cdot \frac{\cancel{|AP|}}{\cancel{|BQ|}} \cdot \frac{\cancel{|BQ|}}{\cancel{|CR|}} \cdot \frac{\cancel{|CR|}}{\cancel{|AP|}} = -1.$$

$\boxed{\Leftarrow}$ Suppose $\frac{AJ}{JB}\cdot\frac{BL}{LC}\cdot\frac{CK}{KA} = -1$. Let L' be the intersection of JK and BC. By $\boxed{\Rightarrow}$, it follows that $\frac{AJ}{JB}\cdot\frac{BL'}{L'C}\cdot\frac{CK}{KA} = -1$, and hence $\frac{BL}{LC} = \frac{BL'}{L'C}$.
Hence $L = L'$ & so $J, K,$ & L are collinear. //

Menelaus (c 100 A.D.) gave this theorem in his only surviving work, Sphaerica, which survived only in Arabic translation. (Apparently Ceva also rediscovered this one.)