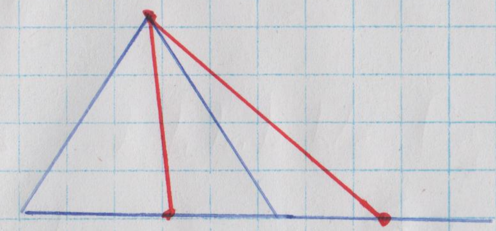
MATH 2260H Ceva's Theorem

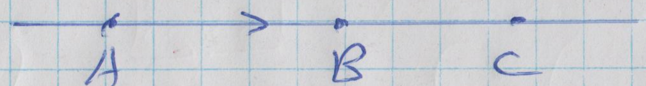
Def'n: A line connecting a vertex of a triangle to (an extension of) the opposite side is called a <u>cevian</u>.

cs



Def'n: Suppose we choose a direction, "forwards", in a line. Then a segment of the line, AB, has positive length if $A \to B$ is forwards, and negative length if $A \to B$ is backwards.

Notation: $AB = \begin{cases} |AB| & \text{if } A \to B \text{ forwards} \\ -|AB| & \text{-- '' --- backwards} \end{cases}$



Note that (1) $AB = -BA$

(2) $\frac{AB}{BC} > 0$ if B is between A & C

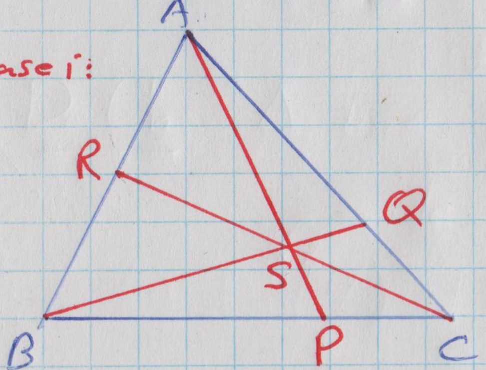
$\frac{AB}{BC} < 0$ if B is not between A & C.

Ceva's Thm: Suppose AP, BQ, & CR are cevians of ΔABC.

[So P is on BC, Q is on AC, & R is on AB.]

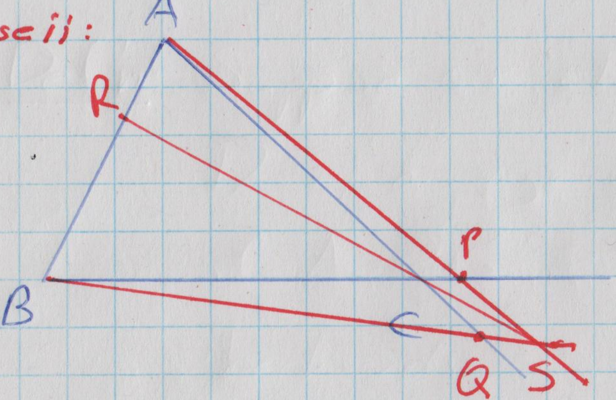
Then AP, BQ, & CR are concurrent in some point S iff $\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = +1$.

case i:



case i:	+	+	+
case ii:	+	−	−

case ii:



proof: (case i) $\boxed{\Longrightarrow}$ Suppose AP, BQ, & CR are concurrent in the point S.

Note that ΔACR & ΔBCR have the same height (if we take their bases to be AR & BR).

so $\frac{|AR|}{|BR|} = \frac{\overset{\text{area}}{|\Delta ACR|}}{|\Delta BCR|}$. Similarly, since ΔASR & ΔBSR have the same height, $\frac{|AR|}{|BR|} = \frac{|\Delta ASR|}{|\Delta BSR|}$.

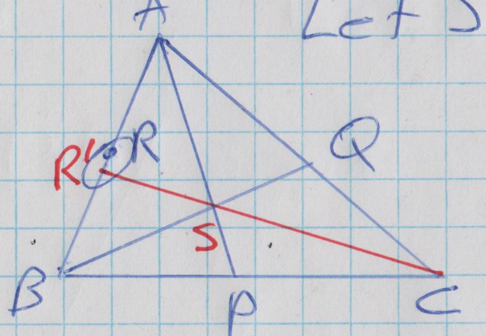
Then $\frac{|AR|}{|RB|} = \frac{|\Delta ACR| - |\Delta ASR|}{|\Delta BCR| - |\Delta BSR|} = \frac{|\Delta ACS|}{|\Delta BCS|}$. Similarly,

we get $\frac{|BP|}{|PC|} = \frac{|\Delta ABS|}{|\Delta ACS|}$ & $\frac{|CQ|}{|QA|} = \frac{|\Delta BCS|}{|\Delta ABS|}$.

But then $\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \left(+\frac{|AR|}{|RB|}\right)\left(+\frac{|BP|}{|PC|}\right)\left(+\frac{|CQ|}{|QA|}\right)$

$$= \frac{|AR|}{|RB|} \cdot \frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|} = +1.$$

$\boxed{\Leftarrow}$ Suppose $\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = +1$.



Let S be the intersection of AP & BQ. Extend CS past S to R' on AB. By $\Rightarrow$, we get that $\frac{AR'}{R'B} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = +1$.

It follows that $\frac{AR}{RB} = \frac{AR'}{R'B}$, so $R = R'$.

case (ii) A similar argument works, except that ΔACR is a subtriangle of ΔASR, so they reverse roles & similarly for the othe triangles.

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History: This theorem seems to have first been discovered in the late 11^{th} century by Yusuf al-Mutaman ibn Hud (d. 1085). It was rediscovered by Giovanni Ceva in 1678.
(1647-1734)