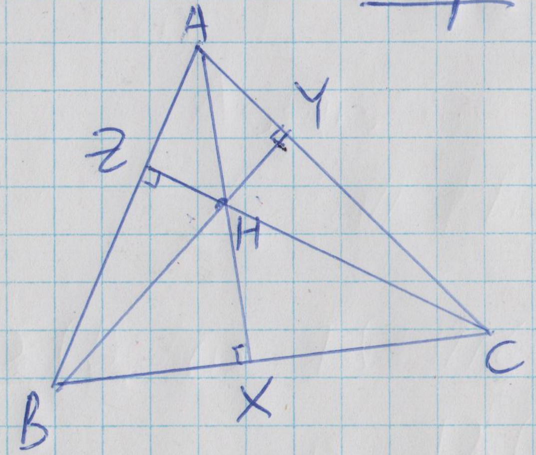
MATH 2260H

Triangles and their centres VI – The orthocentre and the nine-point circle

2021-03-16

Recall: **Prop.:** Suppose H is the orthocentre of ΔABC and X, Y & Z are the feet of the altitudes from A, B, & C, respectively. Then
$$|AH|\cdot|HX| = |BH|\cdot|HY| = |CH|\cdot|HZ|.$$



proof: [Similarity rules again!]

3 cases: (i) H is inside ΔABC (ΔABC is acute)

Left to you! (ii) ΔABC is right (H is one of the vertices) [trivial]

(iii) H is outside ΔABC (ΔABC is obtuse)

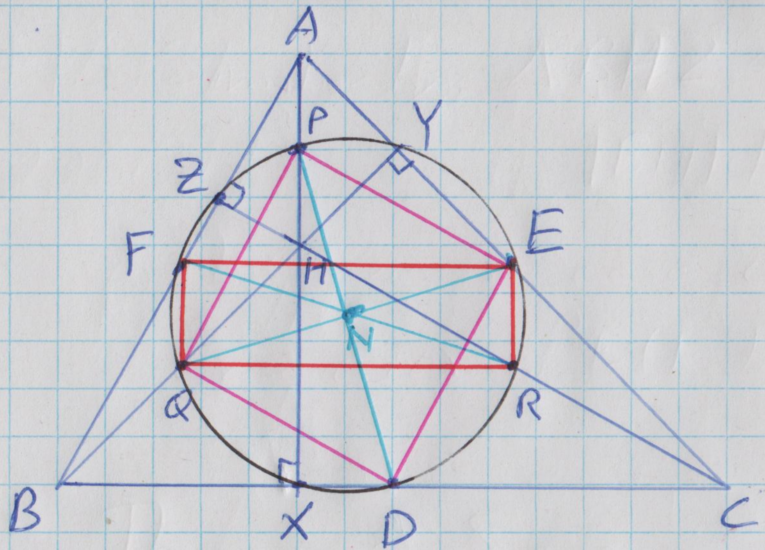
Consider ΔAHY and ΔBHX. $\angle HYA = \llcorner = \angle HXB$ and also $\angle BHX = \angle AHY$ by the opposite angle theorem. By the AA similarity criterion, it follows that $\Delta AHY \sim \Delta BHX$. Thus
$$\frac{|AH|}{|BH|} = \frac{|HY|}{|HX|} = \frac{|AY|}{|BX|},$$
so $|AH|\cdot|HX| = |BH|\cdot|HY|$.

Similarly, $\Delta BHZ \sim \Delta CHY$, so (eventually)

$$|BH| \cdot |HY| = |CH| \cdot |HZ|.$$

This proves case (i). Case (ii) is trivial.
Case (iii) is left to you as an exercise. //

Theorem: Suppose we have a triangle, ΔABC, with D, E, F being the midpoints of sides BC, CA, & AB, respectively, and let X, Y, Z be the feet of the altitudes from A, B, & C, respectively. Let H be the orthocentre of ΔABC and let P, Q, R be the midpoints of AH, BH, & CH, respectively. Then $D, E, F, X, Y, Z, P, Q,$ & R are all on the same circle. [The nine-point circle of ΔABC.]



proof: E & F are the midpoints of sides AB & AC of ΔABC, so $FE \parallel BC$ and $|BC| = 2|FE|$.

Q & R are the midpoints of sides BH and CH of ΔHBC, so $QR \parallel BC$ and $|BC| = 2|QR|$.

It follows that $|FE| = |QR|$ and $FE \parallel QR$. (So FQRE is a parallelogram)

Since F & Q are the midpoints of the sides AB and HB of ΔHAB, we have $FQ \parallel AH$ and $|AH| = 2|FQ|$; similarly, since E & R are the midpoints of the sides of ΔHAC, we have $ER \parallel AH$ & $|AH| = 2|ER|$.

It follow that $|FQ| = |ER|$ & $FQ \parallel ER \parallel AH \parallel AX$ which is perpendicular to BC and hence to FE and QR. Thus FQRE is a rectangle. The diagonals of a rectangle intersect in a point that is equidistant from the four vertizes and hence is the centre of a circle passing through all four vertices.

We'll call the intersection of QE and FR, N.

E & D are the midpoints of sides AC and AB of ΔABC and P & Q are the midpoints of sides AH & BH of ΔABC. A similar argument to the one we went through for E, F, Q, R, shows that $EDQP$ is also a rectangle. This rectangle has diagonals that meet in their middles too, and since the two rectangles $FQRE$ & $EDQP$ share the diagonal QE, whose midpoint is N, the six points D, E, F, P, Q, R are all on the same circle.

Why are X, Y, Z on the same circle?

Observe that PD is a diagonal of the circle above, and $\angle PXD = \llcorner = \angle AXD$. By the converse of Thales' Thm., it follows that X is also on the circle. Similar arguments show that Y & Z are also on the circle. //

Note: It turns out that N is also on the Euler line, and is the midpoint of GO.