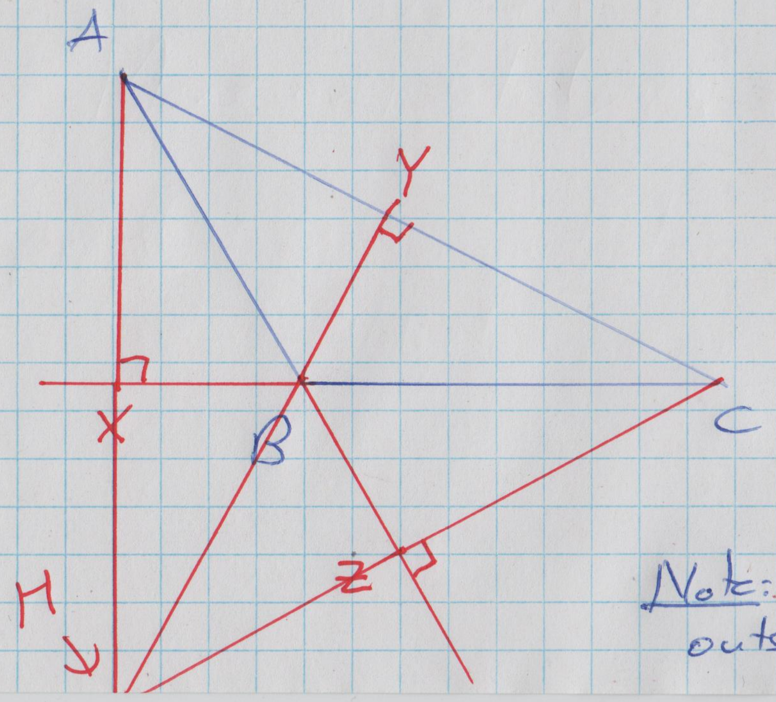
MATH 2260H Triangles and their centres IV – The altitudes and the orthocentre.

So far we have three triangle centres: circumcentre (O), incentre (I), & centroid (G).

<u>Defn</u>: The <u>altitudes</u> of a triangle are the lines from each vertex that are perpendicular to the (extensions of the) opposite sides. The points where the altitudes meet the opposite side are the <u>feet</u> of the altitudes. We'll be denoting the feet by X, Y, Z (no real standard) for the altitudes from A, B, & C respectively.

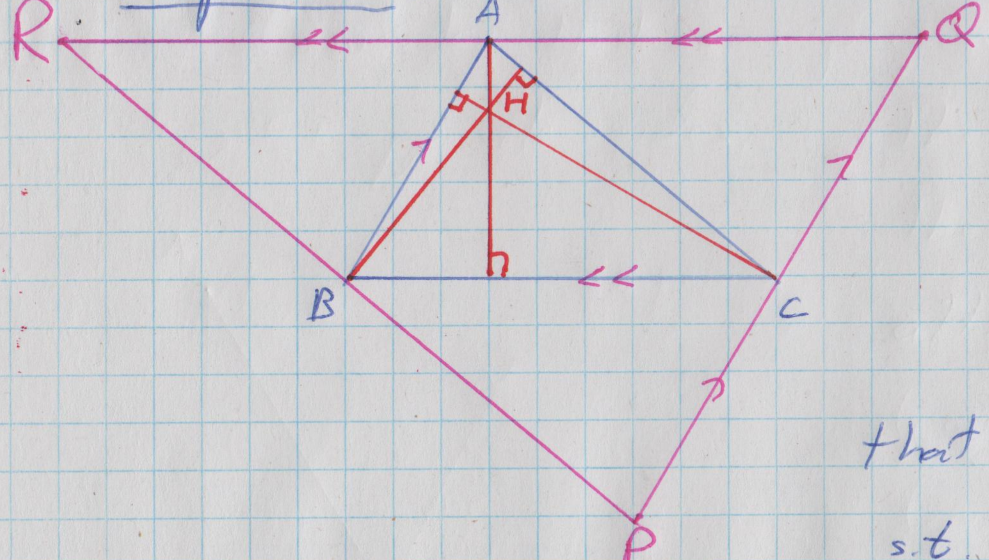
The point where the altitudes meet is the orthocentre, H. (standard)



<u>Note</u>: As with the circumcentre, the orthocentre is outside of $\triangle ABC$ if the triangle is obtuse.

Note: Altitudes occur in Euclid and later authors, but no one seems to have considered the orthocentre until the 1700's. Earliest proof that the altitudes are concurrent (that we know of) seems to be on by William Chapple in 1749).

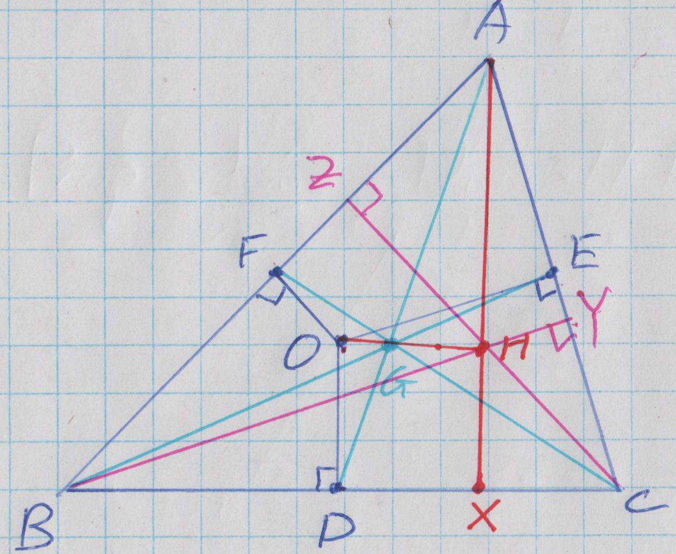
Proposition: The altitudes of a triangle are concurrent.



proof: Given ΔABC draw a line segment PQ parallel to AB with midpoint C and $|PQ| = 2|AB|$. Similarly, draw $QR \parallel BC$ through A ~~B~~ such that $|AQ| = |QR|$, and $RP \parallel AC$ through B s.t. $|RB| = |BP|$. Note that the perpendicular bisectors of the sides of ΔPQR meet in the circumcentre of ΔPQR, but these perpendicular bisectors are the altitudes of ΔABC, so they are concurrent. //

<u>Corollary:</u> (to the proof) Let D, E, F be the midpoints of sides BC, CA, AB (resp.) of ΔABC. Then the orthocentre of ΔDEF is the circumcentre of ΔABC.

<u>Alternate proof that the altitudes are concurrent:</u> (Euler)



<u>proof:</u> Let O and G, resp., be the circumcentre and centroid of ΔABC. Join O to G and extend this to a ~~line~~ point H s.t. $|GH| = 2|OG|$. <u>Claim:</u> H is the orthocentre of ΔABC.

Draw AH and extend to X on BC. We will show that AH (& hence AX) is perpendicular to BC.

$\angle OGD = \angle HGA$ by the Opposite Angle Theorem (I-15).

By construction, $|GH| = 2|GO|$. Since the centroid G divides the median AD is a 2:1 ratio, we have $|GA| = 2|GD|$. By the SAS similarity criterion, $\Delta OGD \sim \Delta HGA$, so $\angle GDO = \angle GAH$.

By the Z-thm, it follows that $OD \parallel AH$. Since OD is perpendicular to BC, AH (and hence AX) is perpendicular to BC, too.

Similar arguments show that BH and CH are perpendicular to AC and AB, respectively. Thus the altitudes of $\triangle ABC$ pass through H. //

Corollary (to the proof): The circumcentre O, the centroid G, and the orthocentre H of a triangle are collinear, and G is between O and H with $|GO| = \frac{1}{2}|GH|$.

Def'n: The line on which O, G, & H all are is called the Euler line.

More next time!