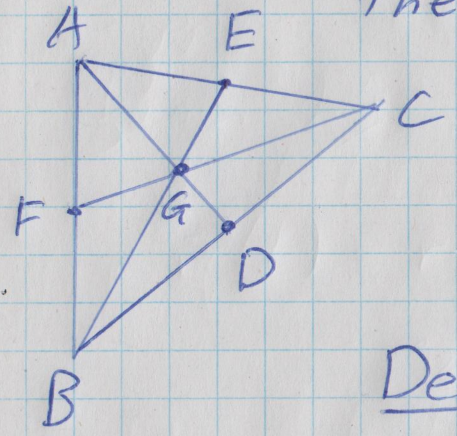
MATH 2260H Triangles and their centres II - The centroid

Defn: The line joining the vertex of a triangle to the midpoint of the opposite site is a median.



Theorem: The three medians of a triangle are concurrent. (ie they meet at a single point)

Defn: The point at which the medians meet is the centroid of the triangle.

Notation: It's traditional to call the centroid G, the circumcentre O, and the incentre I.

Aside:

Encyclopedia of Triangle Centers

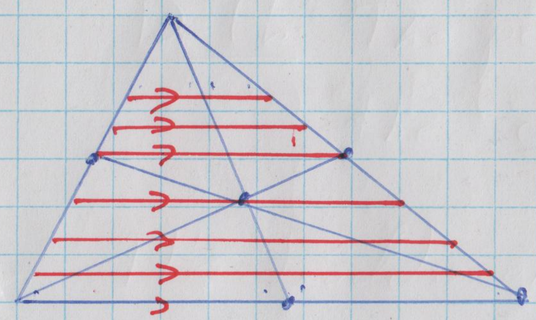
https://faculty.evansville.edu/ck6/encyclopedia/ETC.html

There are thousands of the fool things...

More generally, the centroid of a 2-D figure is its centre of mass if it has uniform composition.

eg The centroids of a circle and a square, respectively, are the centre of the circle and the intersection point of the diagonals of the square.

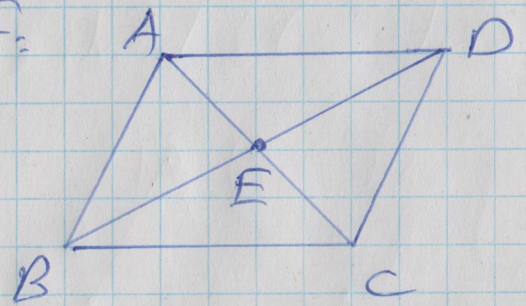
The centroid of a triangle is also its centre of mass.



Consider the lines ~~parallel to~~ ~~one of the medi~~ parallel to one of the sides of the triangle. Using similarity it is not hard to see that the median that meets that side at its midpoint divides each of these cross-sections of the triangle in half, so each cross-section will balance on the median, so the triangle as a whole will balance on the median. Since this is true for all three medians, the point where they meet must be the balance point for the entire triangle.

We'll need a couple of Lemmas to give a proper argument for why the medians are concurrent.

<u>Lemma:</u> The diagonals of a parallelogram bisect each other.

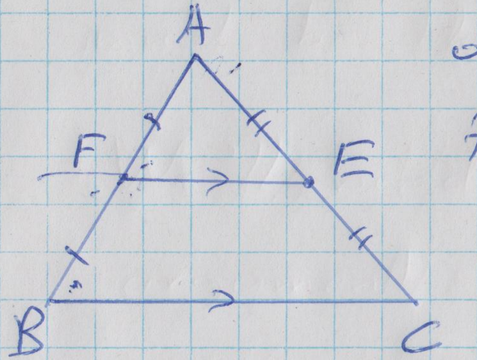
proof: 

Let E be the intersection of the diagonals AC and BD of parallelogram $ABCD$. Consider the triangles ΔAED and ΔCEB. By the Z-theorem, $\angle DAE = \angle DAC = \angle BCA = \angle BCE$, and also

$$\angle ADE = \angle ADB = \angle CBD = \angle CBE.$$

Since $ABCD$ is a parallelogram $|AD| = |CB|$. By the ASA congruence criterion, $\Delta AED \cong \Delta CEB$, but then $|AE| = |CE|$ and $|BE| = |DE|$, so E is the midpoint of both diagonals, so they bisect each other. //

Lemma: The line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.



proof: Let F and E be the midpoints of sides AB and AC, respectively, in ΔABC.

Then $\cancel{\angle AFE}$ $\angle FAE = \angle BAC$ [same angle] and $|BA| = 2|FA|$ and $|CA| = 2|EA|$, so by the SAS similarity criterion, $\Delta FAE \sim \Delta BAC$. It follows that

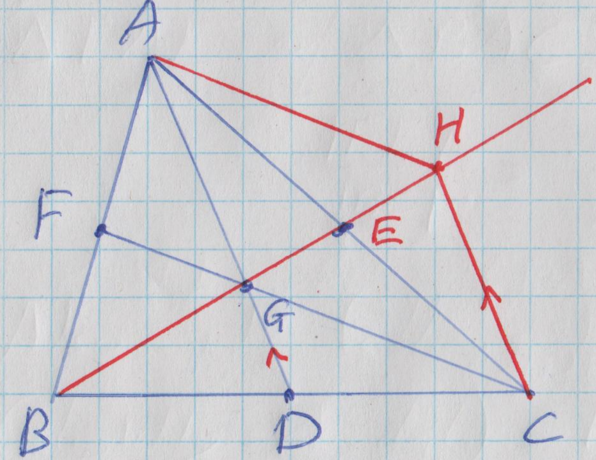
$$\frac{|FE|}{|BC|} = \frac{|FA|}{|BA|} = \frac{1}{2}, \text{ so } |BC| = 2|FE| \text{ i.e. } |FE| = \tfrac{1}{2}|BC|.$$

Also, it follows that $\angle AFE = \angle ABC$, so, by the Z-thm. (slightly extended) $FE \parallel BC$. //

<u>Thm:</u> The medians of a triangle are concurrent in a single point.

<u>Note:</u> The proof we are giving is a minor variation on a proof devised by a high school teacher, Dani Rubinstein, and appeared in <u>Mathematics Teacher</u> in September 2003 issue.

<u>proof:</u>



Let D and F be the midpoints of BC and AB, respectively, in ΔABC. Let G be the intersection of AD and CF.

<u>To show:</u> G is on the median from B.

Draw BG and extend it past AC, meeting AC at E. It will be enough to show that E is the midpoint of AC.

Draw a line through C parallel to the median AD, meeting BG at H.

Consider ΔBDG and ΔBCH. Since $\angle GBD = \angle HBC$ [same angle] and $\angle GDB = \angle HCB$ since $GD \parallel HC$ by the Z-thm., the triangles must be similar, $\Delta BDG \sim \Delta BCH$, by the AA similarity criterion. Then $\frac{|BG|}{|BH|} = \frac{|BD|}{|BC|} = \frac{1}{2}$, so G is the midpoint of BH.

Join A to H. Consider ΔABH. F is the midpoint of AB and G is the midpoint of BH, so by one of our lemmas, $GF \parallel HA$ and $|GF| = \frac{1}{2}|HA|$. It follows that $CG \parallel HA$ (since CG is a continuation of GF). We already have that $AG \parallel HC$, so $AGCH$ is a parallelogram and GH and AC are it's diagonals. By our other lemma, it follows that their point of intersection, E, is the midpoint of AC and of GH. Thus BE is the third median of the triangle, so all three are concurrent at G. Note also that $|GH| = |BG|$ & $|GE| = |EH|$, so $|BG| = 2|GE|$. //

Corollary to the proof: The centroid of a triangle is two thirds of the way along each median from the vertex to the midpoint of the opposite side.

More next time!