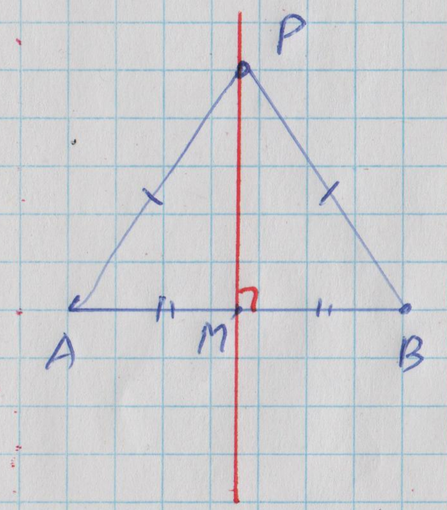
MATH 2260H Triangles and their centres

... starting with two that have to do with circles related to the triangle.

__Lemma:__ A point P is on the perpendicular bisector of AB if and only if it is equidistant from A and B.



__proof:__ $\boxed{\Rightarrow}$ Suppose P is on the perpendicular bisector of AB. Then $\angle PMA = \angle PMB$, since both are right angles, $|AM| = |BM|$ since M is the midpoint of AB, and $|PM| = |PM|$. So $\Delta PMA \cong \Delta PMB$ by SAS criterion, but then $|PA| = |PB|$, so P is equidistant from A and B.

$\boxed{\Leftarrow}$ Suppose P is equidistant from A and B. Then $|PA| = |PB|$, $|AM| = |BM|$ (as above), & $|PM| = |PM|$, so $\Delta PMA \cong \Delta PMB$ by the SSS criterion. Thus $\angle PMA = \angle PMB$ and $\angle PMA + \angle PMB = \angle AMB = \pi$, so $\angle PMA = \angle PMB = \frac{\pi}{2}$. Thus P is on the perp. bisector //

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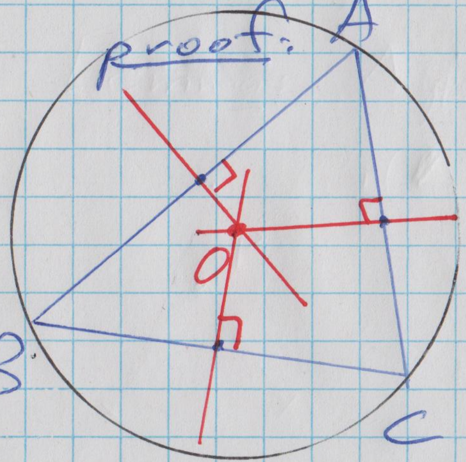
<u>Proposition:</u> The perpendicular bisectors of ΔABC are concurrent at a point O.

- This point is the circumcentre of ΔABC.

<u>proof:</u>



Let O be the point where the perpendicular bisectors of AB and AC meet. O is equidistant from A & B because it is on the perp. bisector of AB and equidistant from A & C because it is on the perp. bisector of AC. } by $\Rightarrow$ of the Lemma

Then O is also equidistant from B and C $(|OB| = |OA| = |OC|)$, so it is also on the perp. of BC. } by $\Leftarrow$ of the Lemma.

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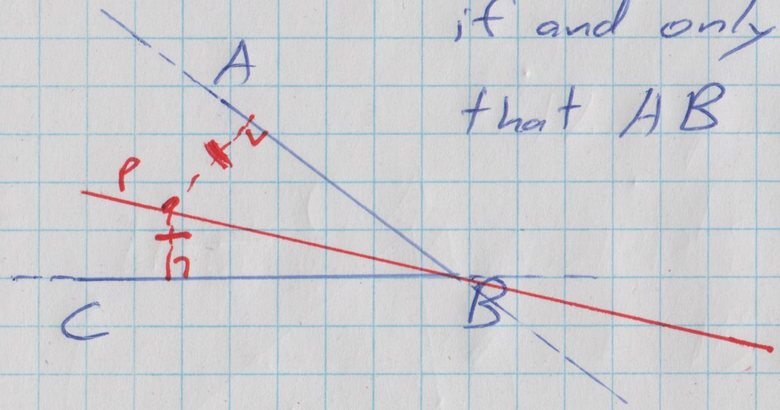
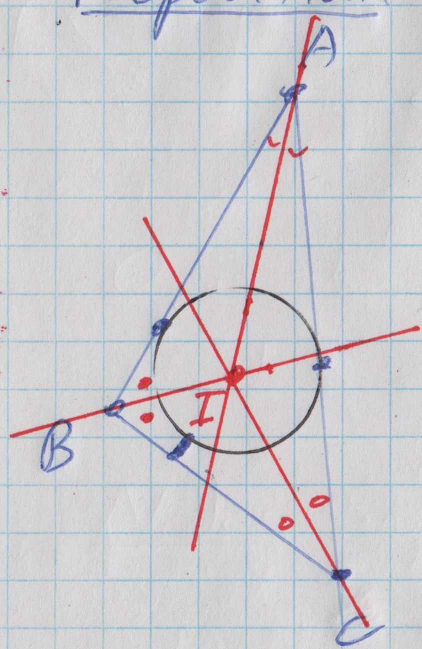
<u>Note:</u> Since O is equidistant from the vertices of ΔABC, a circle with radius OA passes through all three vertices. <u>ie</u> The circumcentre of ΔABC is the centre of the circumcircle of ΔABC.

<u>Lemma:</u> P is on the angle bisector of $\angle ABC$ if and only if P is equidistant from the lines that AB and BC are part of.



<u>proof:</u> Left to you. //

<u>Proposition:</u> The angle bisectors of the internal angles of ΔABC are concurrent in a point I (the <u>incentre</u> of ΔABC).



<u>proof:</u> Let the angle bisectors of $\angle ABC$ and $\angle BAC$ meet at I. Then I is equidistant from AB and AC by the Lemma $\boxed{\Rightarrow}$ and equidistant from AB and BC by the lemma $\boxed{\Rightarrow}$, so it is equistant from BC and AC ~~by the lemma~~. By the Lemma $\boxed{\Leftarrow}$, it follows that I is on the angle bisector of $\angle ACB$. // (I is the centre of the <u>incircle</u> of ΔABC, tangent to all three sides.)