

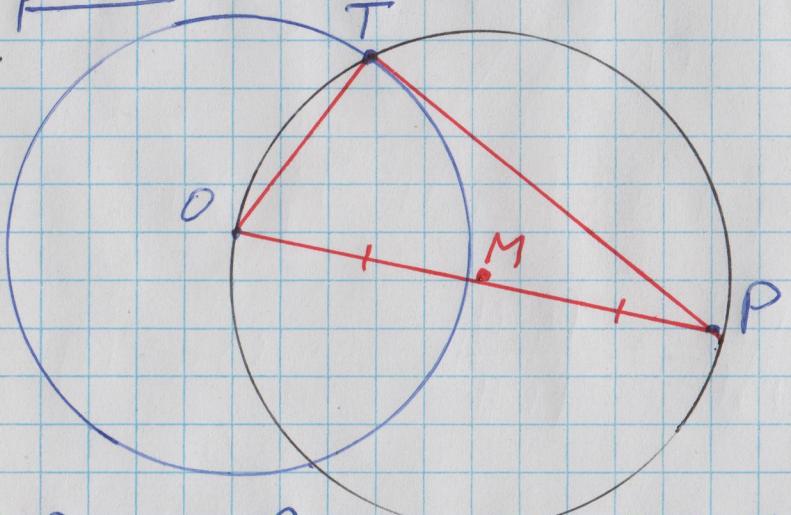
MATH 2260H Circles, Angles, Lines III
or, Apollonian Problems

2021-03-03 ①

Prop. (Somewhere in Book III)

Given a circle and a point outside the circle,
draw ~~a~~^{to the circle} tangent line through the point.

proof: [construction]



Draw OP and let M be the midpoint of OP . Draw a circle with radius MO and centre M . Pick one of the two points of intersection of this circle with the given one and call it T . Draw OT and PT . Claim: PT is tangent to the given circle at T .

Given a circle with centre O
& a point P outside the
circle, how do we draw the
tangent line (either one!) to
the circle ^{and passing} through P .

It's enough to show that OT is perpendicular to PT since OT is a radius of the given circle. (Problem on a recent assignment...). (2)

OP is a diameter of the circle we drew and T is a point on that circle. By Thales' Theorem, it follows that $\angle OTP = 90^\circ$, so OT is perpendicular to PT & so PT is a tangent line to the given circle. //

Problem: (Apollonius of Perga, c. 240 BC - 190 BC.)

Given three objects in the plane, each of which is a point, a line, or a circle, find all the circles that pass through the given point(s), and are tangent to the given line(s) and circle(s).

Apollonius solved this in a lost work called the Tangencies, according to later authors.

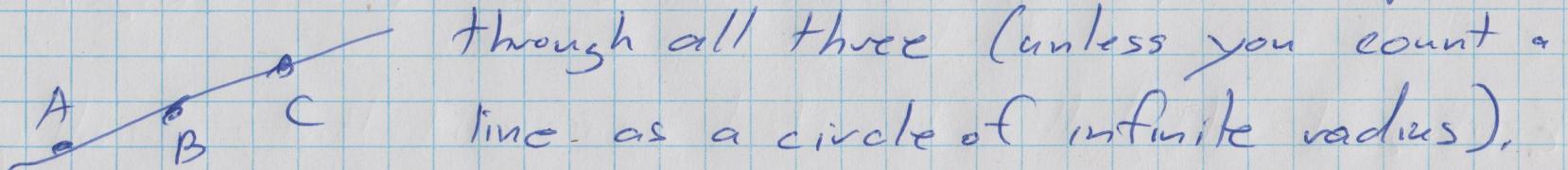
Various people tried to reconstruct this work, based
on what we knew of the problem and his methods
that was related by later authors. The "most likely
to be similar to Apollonius" attempt was by Francois
Vite in his work Apollonius Gallus (1600). ③

We'll take a look at a couple of these problems
and encounter others later.

Problem: (PPP) Given three points find all the circles
that pass through all three points.

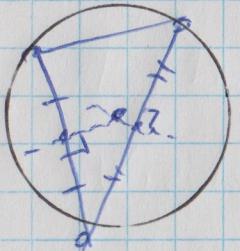
Solution: Suppose we are given three points A, B, C .

2 cases: (i) A, B, C are collinear; no circle passing



through all three (unless you count a
line as a circle of infinite radius).

(ii) A, B, C are not collinear, in which case
they form a triangle. (7)

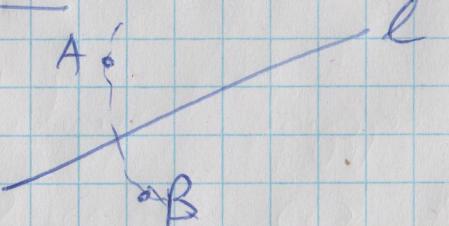


We showed before that there is an unique circle that passes through all three vertices of a given triangle.

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Problem: (LPP) Given a line l and points $A \& B$ (not on l),
find all the circles that pass through $A \& B$
and are tangent to l .

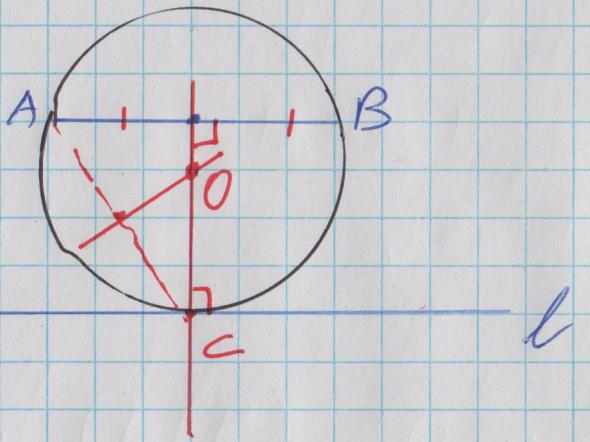
Solution: case (i) $A \& B$ are on opposite sides of l



By Postulate 5 any circle passing
through $A \& B$ will cut l , so
in this case there is no solution.

case (2): A & B are on the same side of l
and $AB \parallel l$.

(5)



Draw the perpendicular bisector
of AB and extend it to
meet l at C .

Draw the circle that passes
through $A, B, \& C$. This

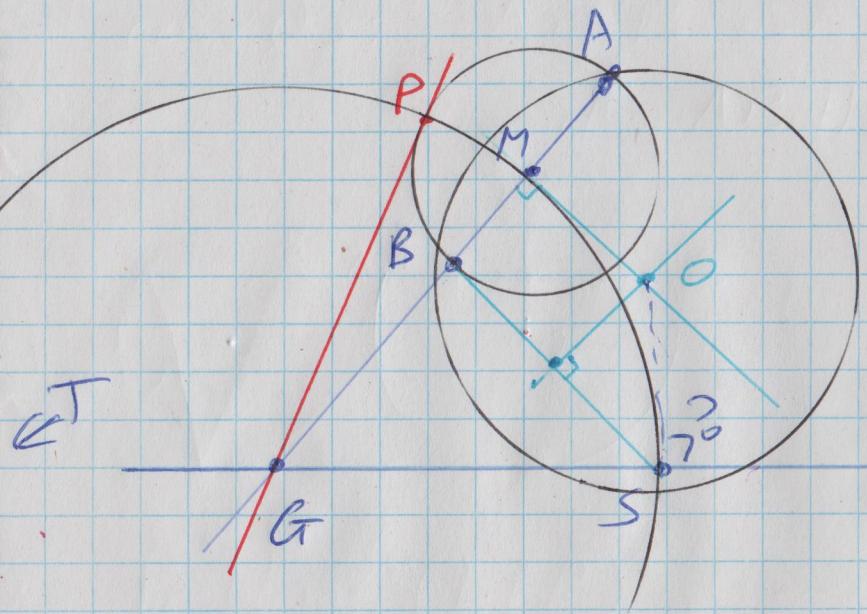
will be tangent to l at C since the perpendicular
bisector of AB will be perpendicular to l and its
an (extension) of a diameter of the circle.

case(3): A & B are on the same side of l and
 $AB \nparallel l$.

~~Draw the perpendicular~~ Extend AB until it meets l
at some point G.

(6)

Let M be the midpoint of AB . Draw the circle with centre M and radius MA . Now draw a tangent line to this circle from G , meeting the circle at P .



Draw the circle with radius GP and centre G . This circle will intersect l at two points S & T . There is an unique circle passing through A, B , & S (and another passing through A, B , & T). Claims: this circle is tangent to l at S (& the other is tangent to l at T). This is left to you as a (non-trivial) exercise.

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