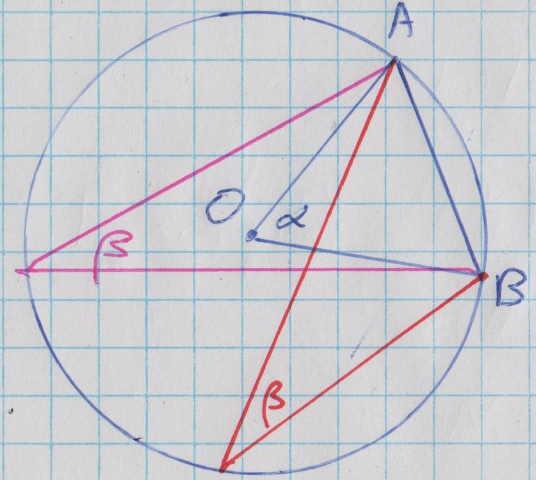
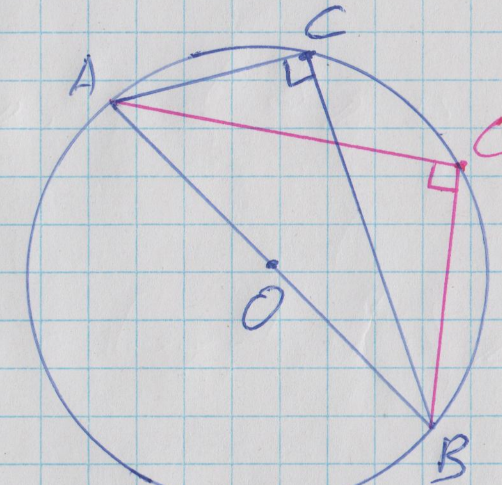
Circles, Angles, Lines II

Recall: (III-20) The angle subtended by a chord from the centre of a circle is twice the angle subtended by the chord from any point on the circle.



$\alpha = 2\beta$

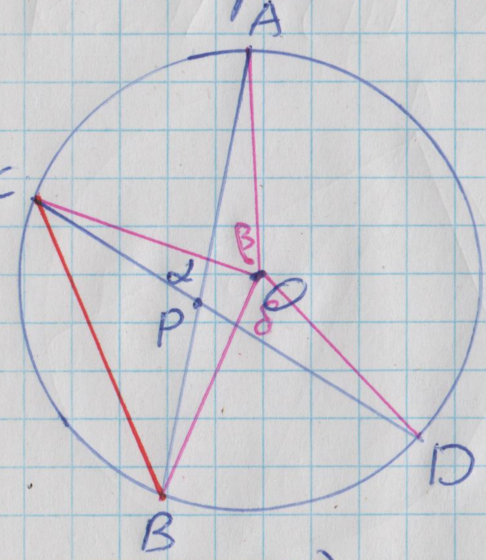
<u>Corollary:</u> The angle subtended by a chord from a point on the circle ~~[illegible]~~ is equal to that subtended from any other point on the circle that is on the same side of the chord.



<u>Thales' Thm:</u> Suppose AB is a diameter of a circle, and C is any other point on the circle. Then $\angle ACB = \llcorner$.

<u>proof:</u> By III-20, $\angle ACB = \frac{1}{2}\angle AOB = \frac{1}{2}\,\underline{\frown} = \llcorner$. //

Prop.: Suppose the chords AB and CD intersect at a point P inside the circle. Then



$$\angle APC = \tfrac{1}{2}\angle AOC + \tfrac{1}{2}\angle BOD.$$

$\alpha = \tfrac{1}{2}(\beta + \delta)$

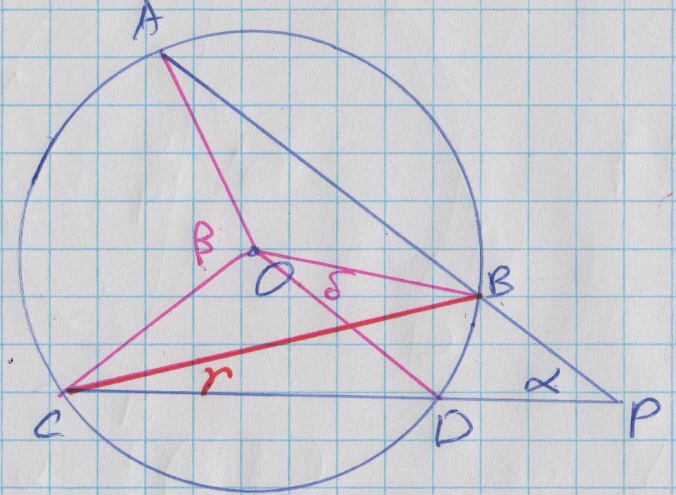
proof: Connect B to C & consider $\triangle BPC$. Then

$$\angle BPC + \angle BCP + \angle PBC = \frown = 2\llcorner = \pi \text{ rad},$$

so
$$\begin{aligned} \angle BPC &= 2\llcorner - \angle BCP - \angle PBC \\ &= 2\llcorner - \angle BCD - \angle ABC \qquad (\text{III-20}) \\ &= 2\llcorner - \tfrac{1}{2}\angle BOD - \tfrac{1}{2}\angle AOC \end{aligned}$$

$$\begin{aligned} \angle APC = 2\llcorner - \angle BPC &= 2\llcorner - \left(2\llcorner - \tfrac{1}{2}\angle BOD - \tfrac{1}{2}\angle AOC\right) \\ &= \tfrac{1}{2}\angle BOD + \tfrac{1}{2}\angle AOC. \qquad // \end{aligned}$$

<u>Prop:</u> Suppose the chords AB and CD have the point of intersection (of their extensions) at P.
Then $\angle APC = \frac{1}{2}\angle AOC - \frac{1}{2}\angle BOD$.



$\alpha = \frac{1}{2}\beta - \frac{1}{2}\delta$

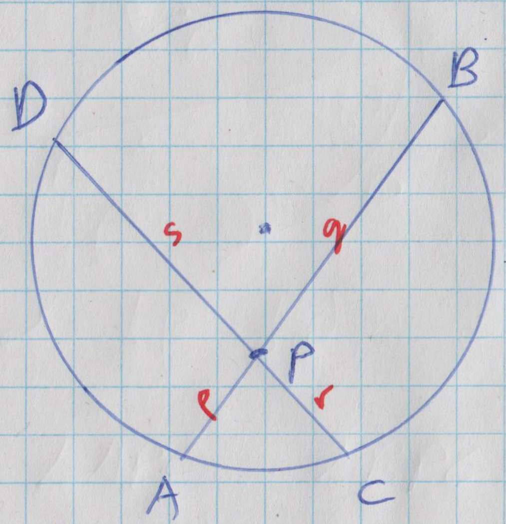
<u>proof:</u> Connect B to C and consider $\triangle BPC$. Then $\angle BPC + \angle BCP = \angle ABC$ since an exterior angle of a triangle is the sum of the opposite interior angles. It follows that

$$\angle APC = \angle BPC = \angle ABC - \underbrace{\angle BCP}_{=\angle BCD}$$
$$= \frac{1}{2}\angle AOC - \frac{1}{2}\angle BOD \qquad \text{(III-20)}$$

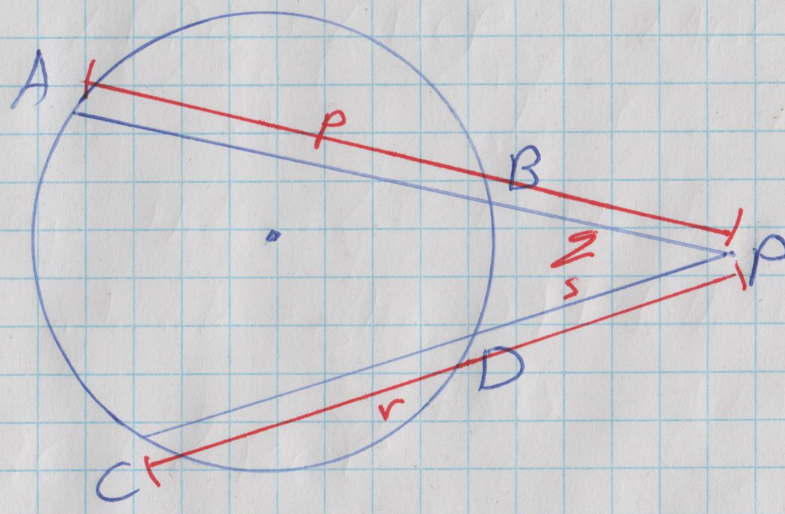
//

We also have results related to these about the "cross-ratios" of chords that intersect each other.

<u>Prop:</u> Suppose AB and CD are chords of a circle that intersect at some point P not on the circle. Then $|PA| \cdot |PB| = |PC| \cdot |PD|$.



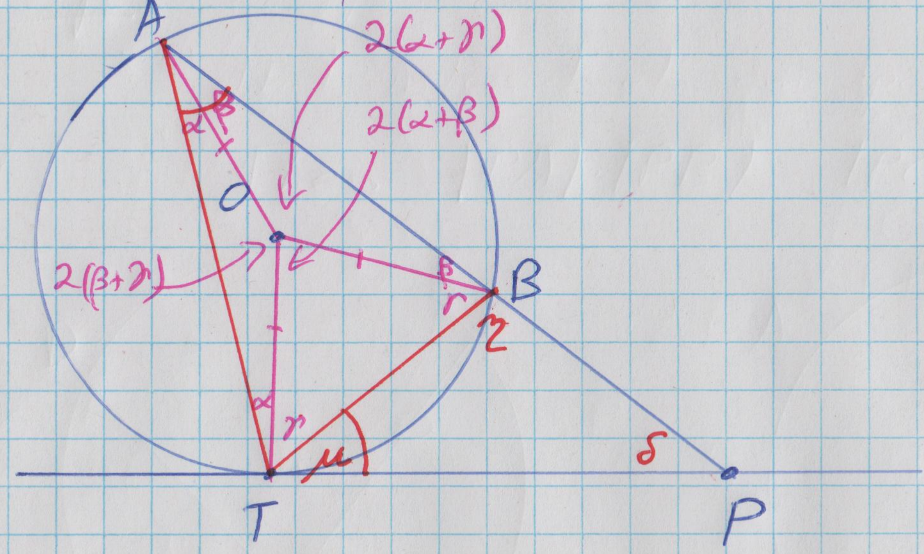
$pq = rs$



$pq = rs$

<u>proof:</u> Left to you! //

Prop: Suppose AB is a chord of a circle with centre O, and (the extension of) AB meets the tangent line at T on the circle at P. Then $|PA|\cdot|PB| = |PT|^2$.



proof: We will draw all the lines and angle we can and see if any are useful. Label the angles as in the diagram, using isosceles triangles having equal angles at the base, III-20, and the angle relations for triangles & angles, ...

$\beta + \gamma + \eta = 2\llcorner$ $\quad \gamma + \mu = \llcorner$ (by a result on A#5)

$2\alpha + 2\beta + 2\gamma = 2\llcorner \Rightarrow \alpha + \beta + \gamma = \llcorner$

What results & ideas relate angles to lengths of sides? Similarity...

Rearrange $|PA|\cdot|PB| = |PT|^2 \Rightarrow \dfrac{|PA|}{|PT|} = \dfrac{|PT|}{|PB|}$ Which suggest that may be we should try for $\Delta PBT \sim \Delta PTA$.

Note that $\angle TBP = \angle APT$ [same angle].

Can we get another angle from each triangle to be equal?

<u>Ie:</u> Get $\alpha+\beta=\mu$ or $\eta = \alpha+\gamma+\mu$?

We know $\mu+\gamma = \pi$ and $\gamma = \pi-\alpha-\beta$

$$\mu = \pi - \gamma = \pi - (\pi-\alpha-\beta) = \alpha+\beta$$

<u>ie</u> $\angle PAT = \alpha+\beta = \mu = \angle PTB \longrightarrow$ Yes!

It follows that $\angle PBT \sim \angle PTA$, so

$$\frac{|PA|}{|PT|} = \frac{|PT|}{|PB|} = \frac{|AT|}{|TB|}.$$

$$\Longrightarrow |PA|\cdot|PT| = |PT|\cdot|PT| = |PT|^2. \quad //$$