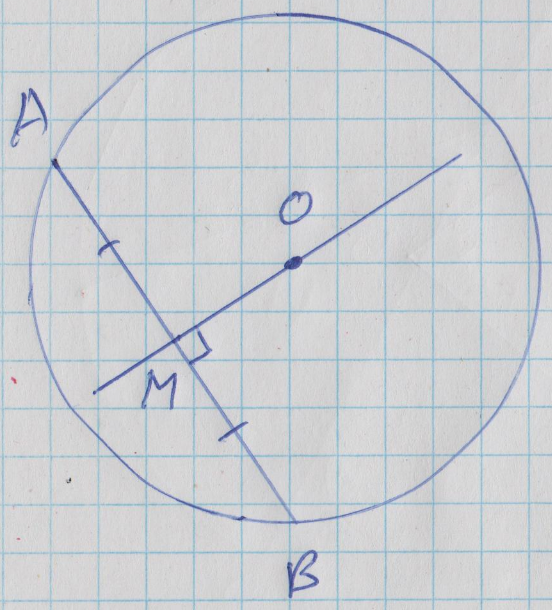
MATH 2260H Circles, Angles, Lines

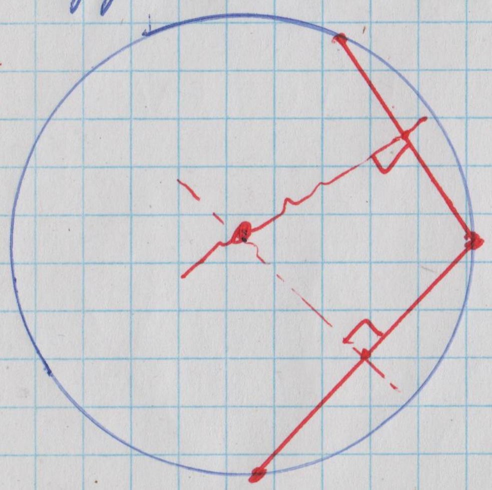
(Bits and pieces from Book III & beyond.)



Proposition: (Corollary to III-1)
The perpendicular bisector of the chord of a circle passes through the centre of the circle.

proof: Left to you! //

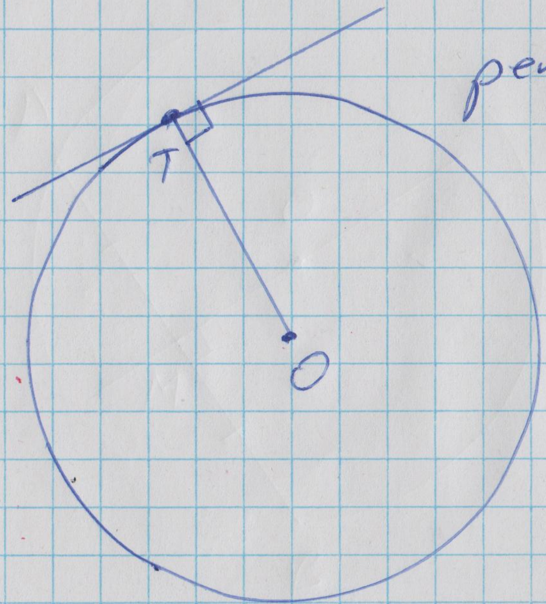
Applications: (1) You can locate the centre of a circle by intersecting the perpendicular ~~of~~ bisectors of different chords.



(2) Any three non-collinear points in the plane determine an unique circle passing through all three points.

proofs: Left to you! //

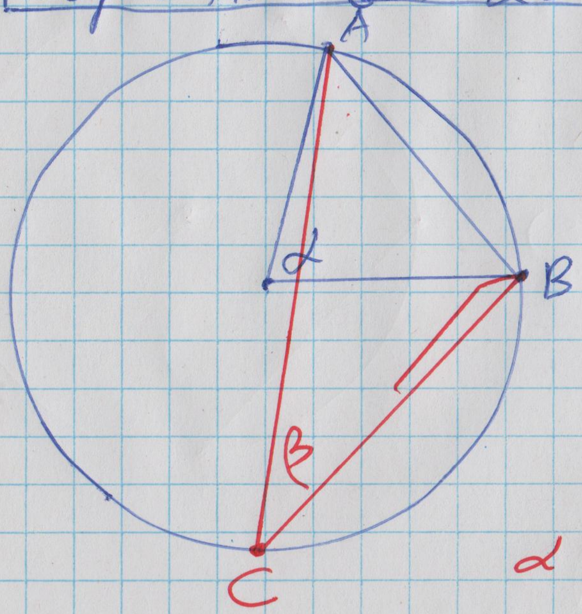
Proposition: Given a circle with centre O and a point T on the circle, a line ℓ through T is perpendicular to OT iff ℓ is tangent to the circle.



proof: Left to the reader! //

(You can look it up - it's the Corollary to III-16 in Book III.)

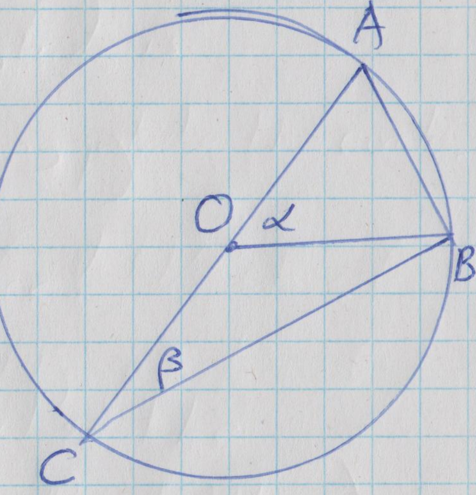
Proposition (III-20): The angle subtended by a chord from the centre of a circle is twice the angle the chord subtends from the circumference of the circle.



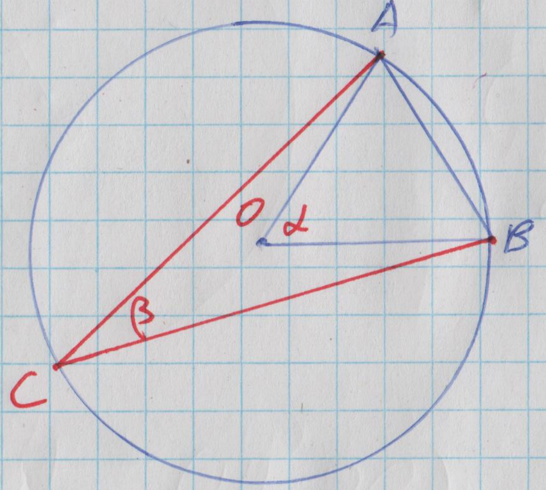
$\alpha = 2\beta$

<u>proof.</u> There are three cases:

(i) the point on the circumference is collinear with the centre of the circle and one endpoint of the chord.

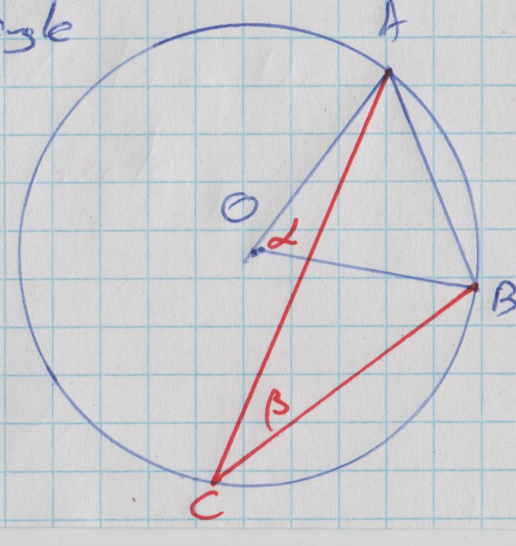


(ii)

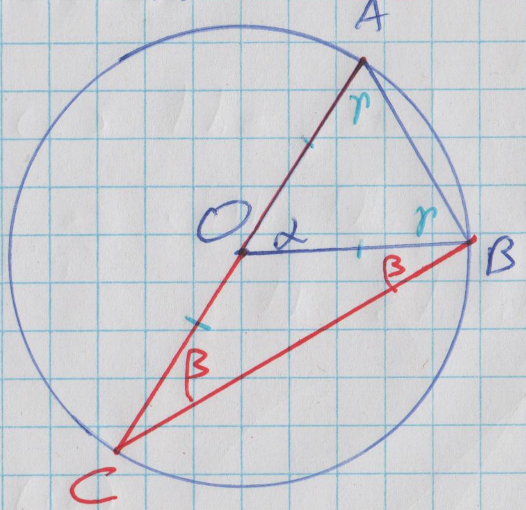


the centre of the circle is inside the angle subtended by the chord from the circumference,

(iii) the centre of the circle is not inside the angle subtended by the chord from the circumference.



case (ii):



Let $\alpha = \angle AOB$ & $\beta = \angle ACB = \angle OCB$.
Since OA, OB, and OC are radii of the circle, ΔAOB and ΔOCB are isosceles. It follows that $\angle OBC = \angle OCB = \beta$, and $\angle OAB = \angle OBA = \gamma$.

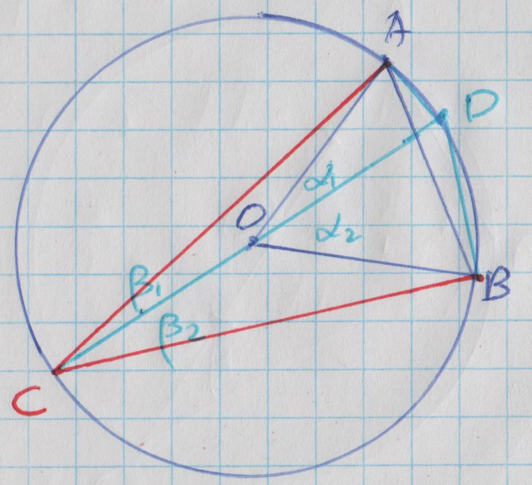
As the sum of the interior angles of a triangle is 2 right angles $= \pi$ radians,

we have $\quad \alpha + \gamma + \gamma = \pi \quad$ (from ΔAOB)

and $\quad \gamma + (\beta + \gamma) + \beta = \pi \quad$ (from ΔABC).

Then $\quad \alpha = \pi - 2\gamma = 2\beta$, so we have $\alpha = 2\beta$, as desired.

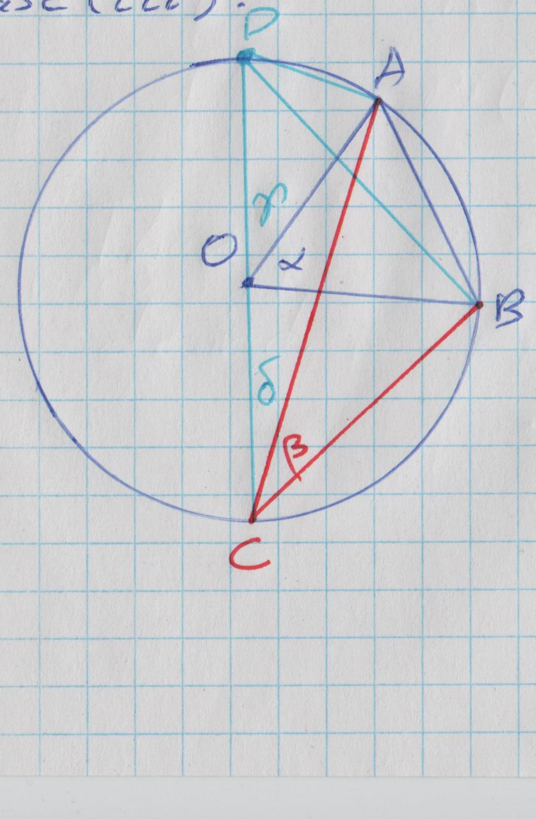
Case (ii):



Connect C to O and extend past O to D on the circle. Let $\angle AOD = \alpha_1$, $\angle BOD = \alpha_2$, $\angle ACD = \beta_1$, and $\angle BCD = \beta_2$. Then by case (i), $\alpha_1 = 2\beta_1$ and $\alpha_2 = 2\beta_2$, so

$\angle ACB = \beta = \beta_1 + \beta_2$ and $\angle AOB = \alpha = \alpha_1 + \alpha_2$ are correctly related: $\alpha = \alpha_1 + \alpha_2 = 2\beta_1 + 2\beta_2 = 2\beta$.

Case (iii):



Connect C to O and extend it past O to D. Let $\angle ACD = \delta$ and $\angle AOD = \gamma$. By case (i), $\gamma = 2\delta$, and also $\alpha + \gamma = 2(\beta + \delta)$, so

$\alpha = 2(\beta + \delta) - \gamma = 2\beta + \cancel{2\delta} - \cancel{2\delta} = 2\beta$,

as desired. //