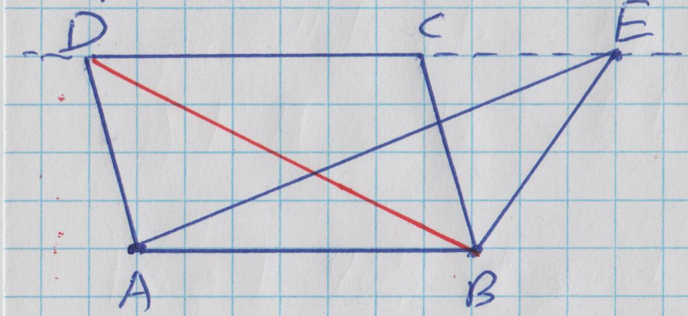
MATH 2260H Propositions I-41 to I-43,

or, more about areas. [of triangles and parallelograms]

Prop I-41: If a parallelogram has the same base as a triangle and is between the same parallels, then the area of the triangle is half the area of the parallelogram.

proof:

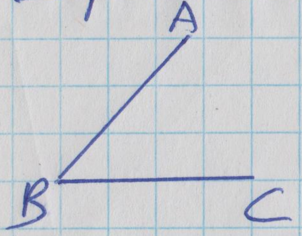


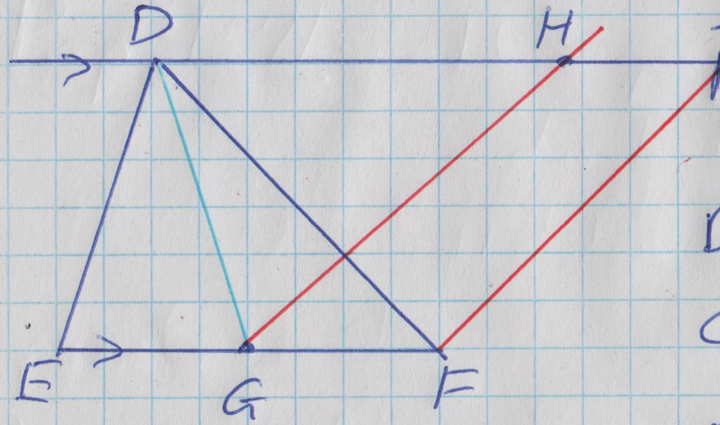
Suppose $ABCD$ is a parallelogram and E is on (an extension of) CD.

To show: $|ABCD| = 2|\triangle ABE|$

Join B and D, creating $\triangle ABD$ which has the same base as $\triangle ABE$ and is between the same parallels. By I-37, it follows that $|\triangle ABD| = |\triangle ABE|$. Since BD is a diagonal of parallelogram $ABCD$, I-34 tells us that $|\triangle ABD| = \frac{1}{2}|ABCD|$, so $|ABCD| = 2|\triangle ABE|$. //

Prop. I-42: Given an angle and a triangle, we can construct a parallelogram which has that angle as an interior angle and area equal to the triangle.



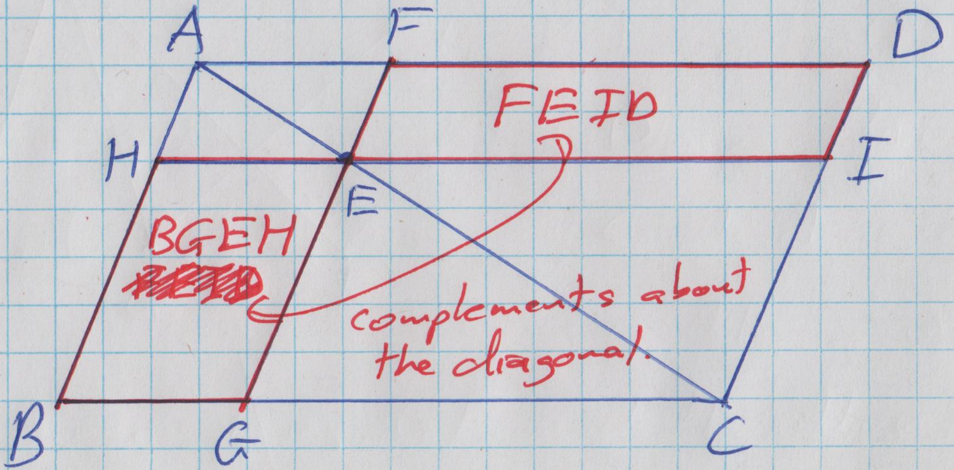


proof: Given $\angle ABC$ and ΔDEF, let G be the midpoint of EF [I-10]. Draw a line through D parallel to EF. Construct an angle equal to $\angle ABC$ at G, on the same side as D [I-23], with one arm being GF and the other meeting the parallel line at H. Draw a line parallel to GH through F, meeting DH (past H...) at I. Then GFIH is a parallelogram with area equal to ~~b~~ twice ΔDGF (join D to G first), since the have the same base and are between the same parallels [I-41]. But $|\Delta DGF| = \frac{1}{2}|\Delta DEF|$,* since ΔDGF has half the base of ΔDEF. Thus $|GFIH| = |\Delta DEF|$. //

*$|\Delta DGF| = |\Delta DEF| - |\Delta DEG|$, but $|\Delta DEG| = |\Delta DGF|$ since they have equal bases & are between the same parallels [I-38].

<u>Prop. I-43</u>: For any parallelogram, the complements of the parallelogram about the diagonal are equal in area.

<u>Q.</u> What are these "complements about the diagonal"?



Given parallelogram ABCD, consider a point E on diagonal AC. Draw a line through E parallel to AB, meeting AD at F and BC at G. Draw a line through E parallel to AD, meeting AB at H and CD at I.

Then parallelograms BGEH and FEID are "complements about the diagonal.

We need to show that $|BGEH| = |FEIH|$.

proof: Note that ΔABC has half the area of $ABCD$, & so does ΔCDA. [Notice the triangles are congruent.]

~~Con~~ Consider parallelogram $AHEF$. In a similar way each of ΔAFE and ΔEHA have the same are.

Thus $|\Delta ABC| - |\Delta AHE| = |\Delta CDA| - |\Delta AFE|$. We also have that $|\Delta ECI| = |\Delta EGC|$ since they are each half of parallelogram $EGCI$. Thus

$$
\begin{aligned}
|FEID| &= |\Delta \overset{ADC}{\cancel{ABC}}| - |\Delta A\overset{F}{\cancel{H}}E| - \cancel{EG}\,|\Delta ECI| \\
&= |\Delta ABC| - |\Delta AHE| - |\Delta EGC| \\
&= |BGEH|.
\end{aligned}
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