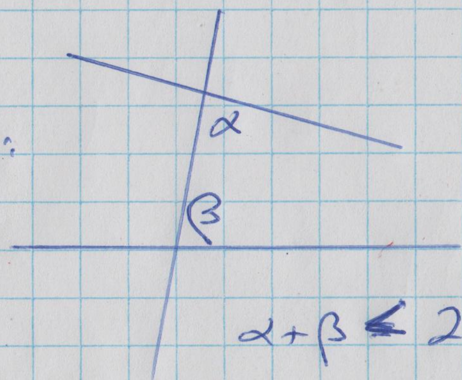
MATH 2260H Some Equivalents of Postulate V, or, too many axioms, or, a break from the *Elements*.

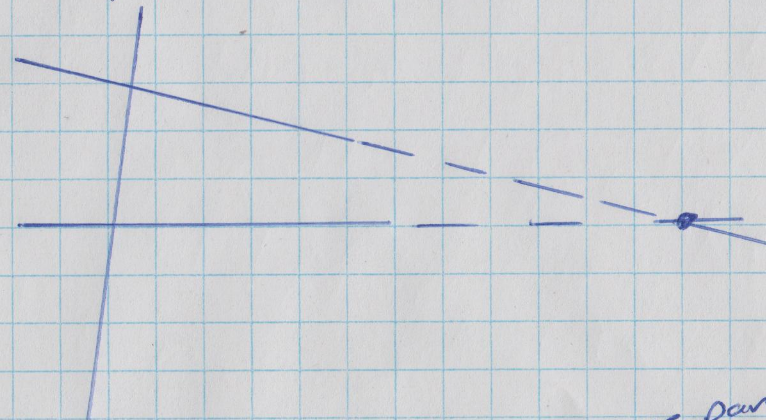
2021-02-05

Post. V:



$\alpha+\beta < 2\llcorner$

$\Longrightarrow$

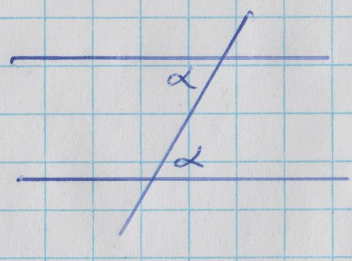


Post. V′: (Playfair's Axiom)

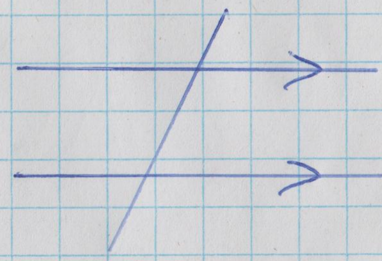
$\Longrightarrow$

unique parallel line to

Post. Z:



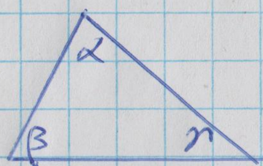
$\Longleftrightarrow$



$\boxed{\Longrightarrow}$ [I-27] only needs Post. I-IV & A&S
does not need Post. V

$\boxed{\Longleftarrow}$ Does need Post. V [I-29] to prove in Euclid's *Elements*

Post. T:



$\Longrightarrow \alpha+\beta+\gamma = 2\llcorner = \measuredangle$

We'll show these 4 are equivalent.

We'll show they are equivalent by proving the following chain of implications [assuming Post. I-IV & A&S].

$$\underline{V} \Rightarrow Z \Rightarrow T \Rightarrow \underline{V}' \Rightarrow \underline{V}$$

$\underline{V} \Rightarrow Z$ is Prop. I-27 & I-29.

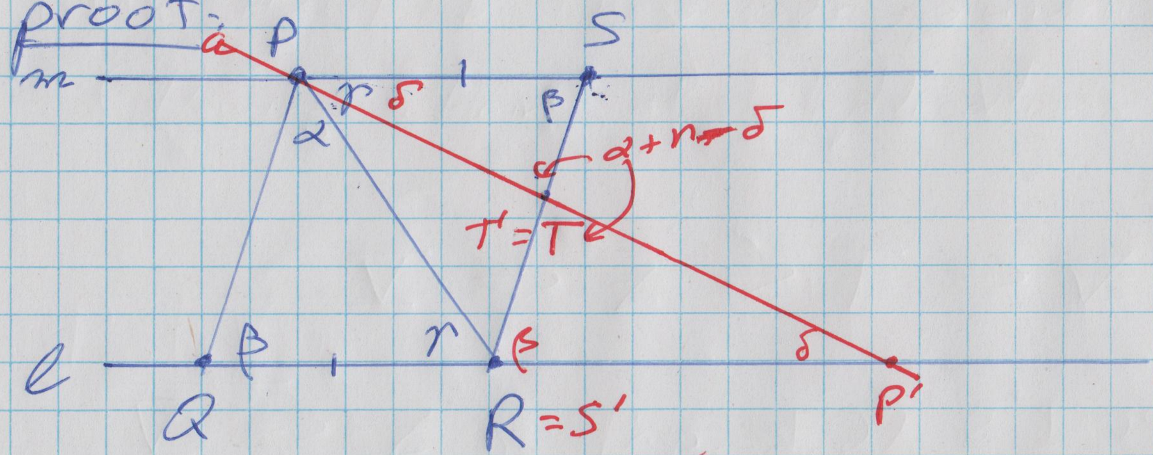
$Z \Rightarrow T$ is Prop. I-32 [does not use Post. V directly].

$T \Rightarrow \underline{V}'$ we need to prove.

$\underline{V}' \Rightarrow \underline{V}$ we also need to prove.

<u>Theorem:</u> [Post I-IV & A&S] Assume that the sum of the interior angles of any triangle is two right angles. Then, given any line ℓ and point P not on ℓ, there is an unique line m through P parallel to ℓ.

proof:



Given P & ℓ with P not on ℓ. Picking two different points Q & R on ℓ, making ΔPQR.

Let $\alpha = \angle QPR$, $\beta = \angle PQR$, & $\gamma = \angle PRQ$.

<u>Existence</u> of a line through P parallel to ℓ:

Make an angle of γ at P with PR as one arm and the other arm being PS on the other side of ~~Q~~ PR from Q and with $|PS| = |QR|$. Let m be the line PS is part of. Then since PR makes equal alternate interior angles of γ with m and ℓ, we have $m \parallel \ell$ by I-27.

<u>Uniqueness</u> of the parallel to ℓ line through P: Suppose a is a line through P which is not equal to m. We need to show that a intersects ℓ.

Join R and S to make $\triangle PRS$.

Let T be the intersection of a with SR.

Since $|RP| = |PR|$ and $\angle SPR = \angle QRP = \gamma$ and $|RQ| = |PS|$,

SAS congruence criterion tells us that $\triangle PQR \cong \triangle PSR$.

$\therefore\ \angle PQR = \beta = \angle PSR$.

Suppose $\angle SPT = \delta$. Then $\angle STP = \underline{\frown} - \beta - \delta$.

Since $\alpha + \beta + \gamma = 2\llcorner$, we have $\beta = 2\llcorner - \alpha - \gamma = \underline{\frown} - \alpha - \gamma$.

Thus $\angle STP = \cancel{\underline{\frown}} - (\cancel{\underline{\frown}} - \alpha - \gamma) - \delta = \alpha + \gamma - \delta$.

By the Opposite Angle Thm [I-15], the angle opposite to $\angle STP$ at T is equal to $\angle STP = \alpha + \gamma - \delta$ too.

Consider $\triangle STP$. Scale this triangle by a factor of $\dfrac{|RT|}{|ST|}$ to get a triangle $\triangle S'T'P'$. Apply $\triangle S'T'P'$ to RT so that $S'T'$ falls on RT (so $S' = R$ & $T' = T$) and P' is on the side of RT opposite to P.

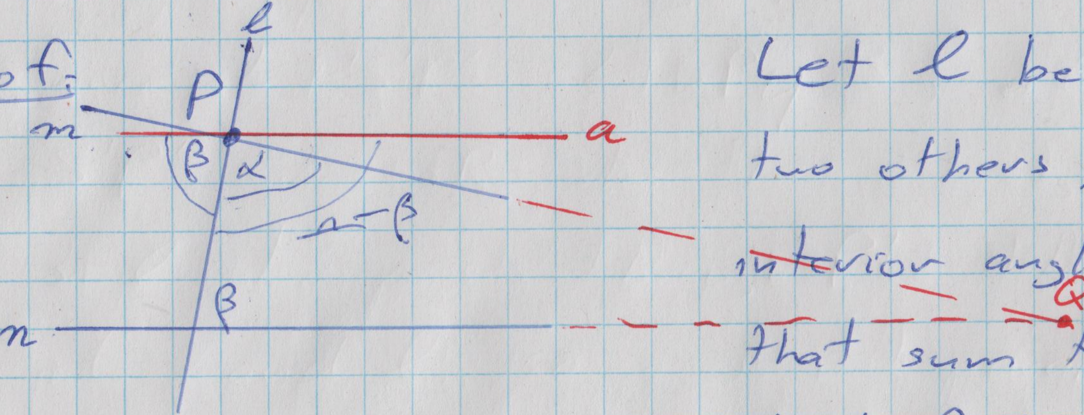
Since $\Delta S'T'P' \sim \Delta STP$ (by it's construction as a scaling of ΔSTP)

we have $\angle S'T'P' = \angle STP = \alpha + \gamma - \delta$. This means that a coincides with $T'P'$ because they make the same angle with SR at T. But this means that a also passes through P'.

Since $\angle T'S'P' = \angle TSP = \beta$, ~~a~~ l ~~also~~ coincides with ~~S'RP'~~ $S'P'$ since SR makes the same angle with both at $R = S'$, so l also passes through P'. Hence P' is the intersection of a and l, so a is not parallel to l, as required. //

It remains to show that Post $\underline{V}'$ $\Rightarrow$ Post $\underline{V}$.

Theorem: Suppose that for any point P and line ℓ not passing through P we have an unique line m through P parallel to ℓ. Then if a line falling across two other lines ~~m~~ makes interior angles adding to less than two right angles on one side, the two lines will intersect on that side.



$\alpha+\beta < 2\llcorner = \measuredangle$

proof: Let ℓ be the line falling across two others, say m & n, making interior angles α & β on one side that sum to less than $2\llcorner$.

Let P be the intersection of ℓ and m.

Draw the line a through P parallel to n by making an angle with ℓ at P equal to β (on the side opposite to α) [I-27 & I-23].

Since $\alpha+\beta < \measuredangle$, we have $\alpha < \measuredangle - \beta$ so a is different from m, so it must intersect ~~m~~ n at some point Q. Since the sum of ~~the~~ any two interior angles of a Δ is $< 2\llcorner$ (I-17), this must on the side with α & β. //