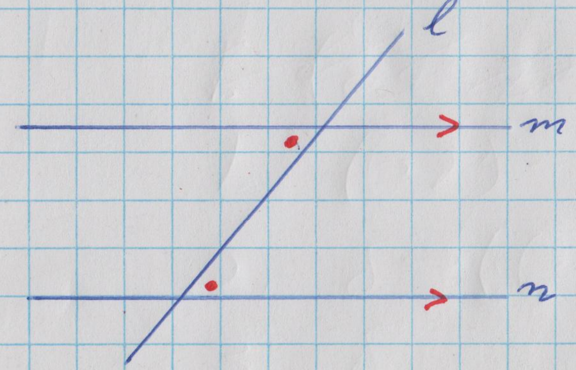
MATH 2260H Propositions I-27 to I-29

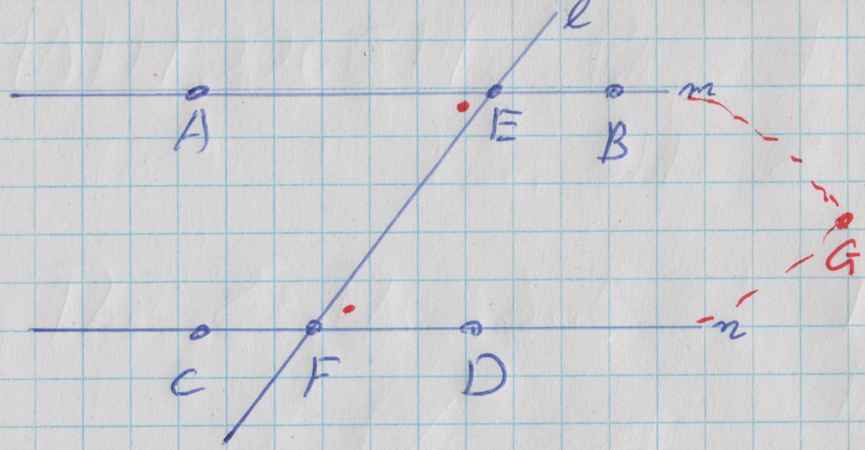
or, the Z-theorem [parallel lines, at last!]

The Z-Theorem: If a line ℓ falls across two other lines m and n, then it makes alternate interior angles equal if and only if m is parallel to n.



[Notation: m is parallel to n is often written as $m \parallel n$. Similarly, m is perpendicular to n is often written as $m \perp n$.]

proof: $\boxed{\Rightarrow}$ [Proposition I-27] Suppose line ℓ falls across m at E, between points A and B on m, and falls across n at F, between points C and D on n.



Suppose further that

$\angle AEF = \angle DFE$.

[To show: $m \parallel n$]

Assume, by way of contradiction that m and n intersect at some point G. [Without loss of generality, we can assume that G is on the same side of l as B and D are. If not, a symmetric argument will work on the other side.]

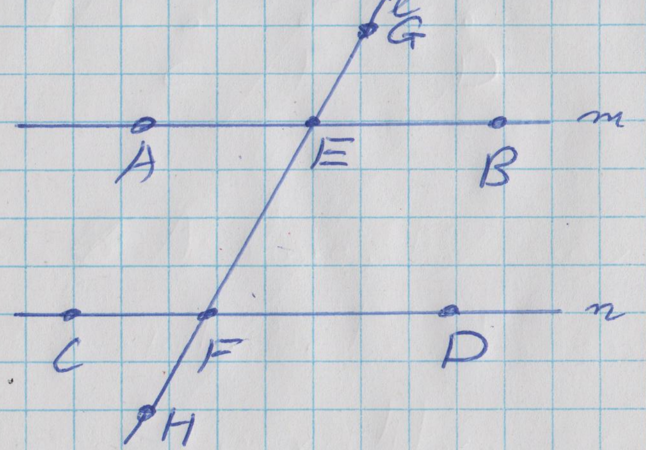
Consider the triangle ΔEFG. $\angle AEF$ is an external angle of that triangle which is equal to the opposite and interior angle of the triangle $\angle DFE = \angle GFE$.

But this contradicts I-16, by which $\angle GFE > \angle AEF$.

$\therefore$ m and n cannot intersect, so $m \parallel n$. //

$\boxed{\Leftarrow}$ [Part of Prop. I-29]

Suppose that ℓ falls across m and n ~~and makes~~ ~~the~~ and $m \parallel n$. [To show: alternate interior angles are equal.]



Suppose we have points on these lines as in the diagram.

We want to show that $\angle AEF = \angle DFE$.

Assume, by way of contradiction, that $\angle AEF > \angle DFE$. [A symmetric argument will take care of $\angle AEF < \angle DFE$.]

Consider the fact that

$$\underbrace{\angle AEF + \angle FEB}_{\substack{\parallel \\ \angle AEB \\ \parallel \\ \text{—}\ = 2\llcorner}} > \angle FEB + \angle DFE .$$

[Take a peek at I-28, which extends I-27 to other angles, & the rest of I-29, which is the converse of I-28.]

Then $\angle FEB + \angle DFE < 2\llcorner$ and they are on the same side of ℓ. By Postulate $\underline{V}$, it follows that m & ~~m~~ n will intersect on that side of ℓ, contradicting $m \parallel n$.

$\therefore \angle AEF = \angle DFE$ by contradiction //.