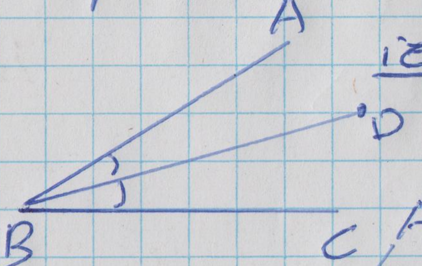
MATH 2260H - Angles & Lines

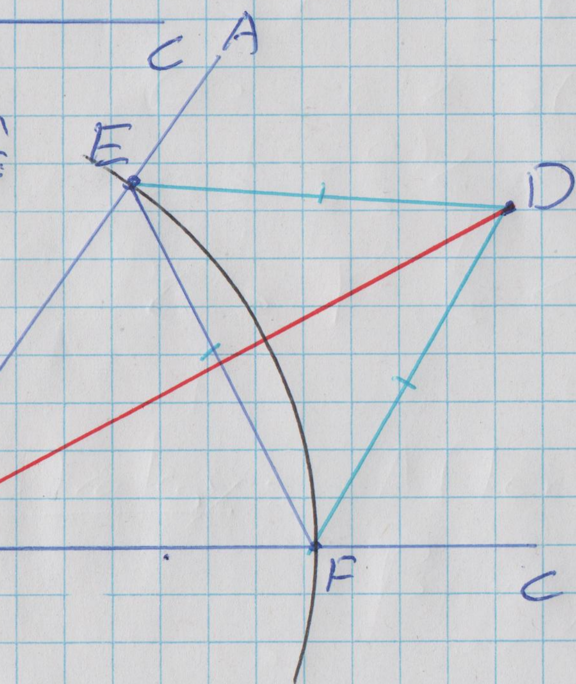
(or, Propositions I-9 to I-12)

Prop. I-9: "To cut a given rectilinear angle in half."



ie Given an angle $\angle ABC$ find a point D s.t. $\angle ABD = \angle CBD$.

proof:



(1) Pick a point E on AB (other than B).

(2) Draw the circle with centre B and radius BE. Call the intersection of the circle with BC, F. (If necessary, extend BC past C.)

(3) Connect E to F.

(4) Construct an equilateral triangle on EF, with the third vertex being D.

(5) Connect B to D.

Claim: $\angle ABD = \angle CBD$

<u>proof of claim:</u> Consider ΔEBD and ΔFBD.

- Since ED and FD are sides of the same equilateral triangle, we have $|ED| = |FD|$.
- Since EB and FB are both radii of the same circle, we have $|EB| = |FB|$.
- $|BD| = |BD|$

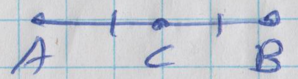
It follows by the S-S-S congruence criterion (Prop. I-8) that $\Delta EBD \cong \Delta FBD$, and thus $\angle EBD = \angle FBD$.

Since $\{\angle ABC = \angle EBF\}$ and so on, we have that

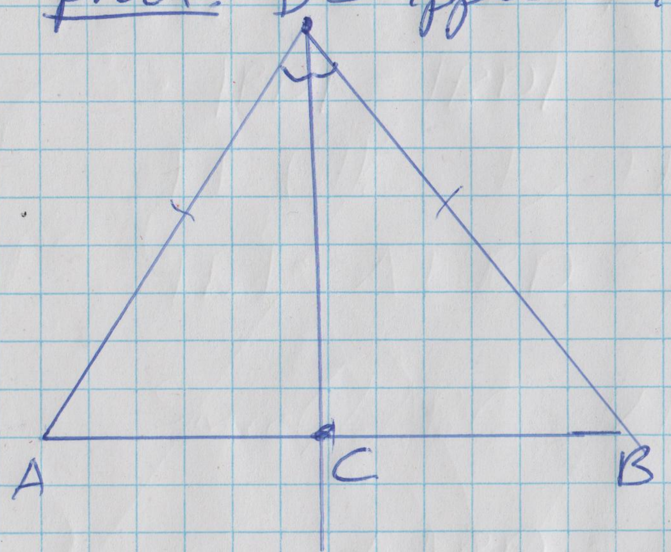
$$\angle ABD = \angle EBD = \angle FBD = \angle CBD,$$ as required.

Prop. I-10: "To cut a given straight line in half."

i.e. Given AB, find a point C such that $|AC| = |CB|$ and C is between A and B on AB.



proof: Suppose AB is given. Construct an equilateral triangle ΔABD on AB.

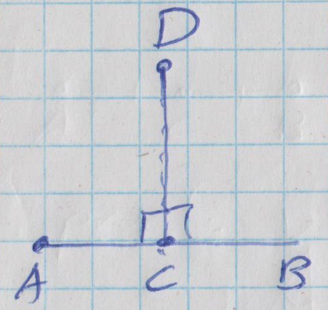


Use Prop. I-9 to divide angle $\angle ADB$ in half, with the dividing line intersecting AB at C.

Then $\angle ADC = \angle BDC$, $|AD| = |BD|$ since they are sides of the same equilateral triangle, and $|DC| = |DC|$, so $\Delta ADC \cong \Delta BDC$ (are congruent), by the SAS congruence criterion.

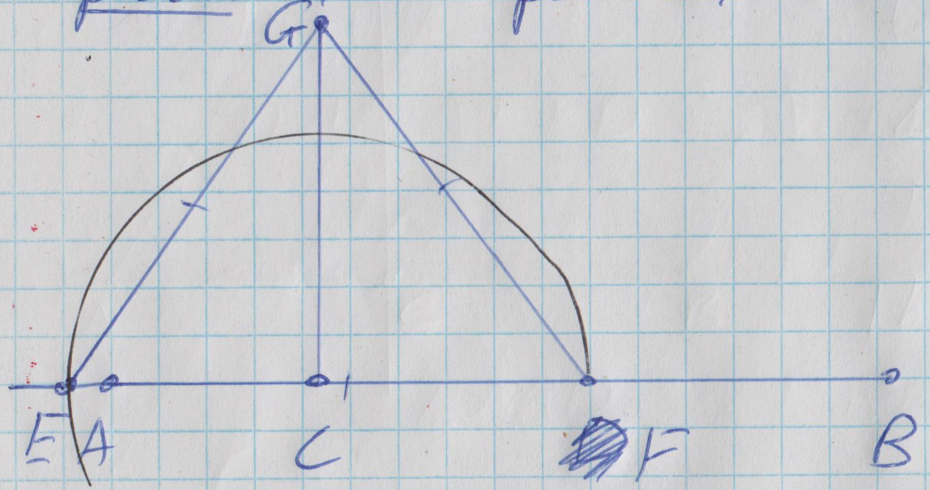
Thus $|AC| = |BC| = |CB|$. //

Prop I-11: "To draw a straight line at right-angles to a given straight line from a given point on it."



ie Given AB and C on AB, find a D (not on AB) such that $\angle ACD$ is a right angle (and $\angle BCD$ too).

proof: Find a point F on AB other than C.



Draw a circle with radius CF and centre C, intersecting AB again at E. (If necessary, extend AB in one direction or the other until it meets the circle.)

Then $|EC| = |FC|$ since both are radii of this circle.

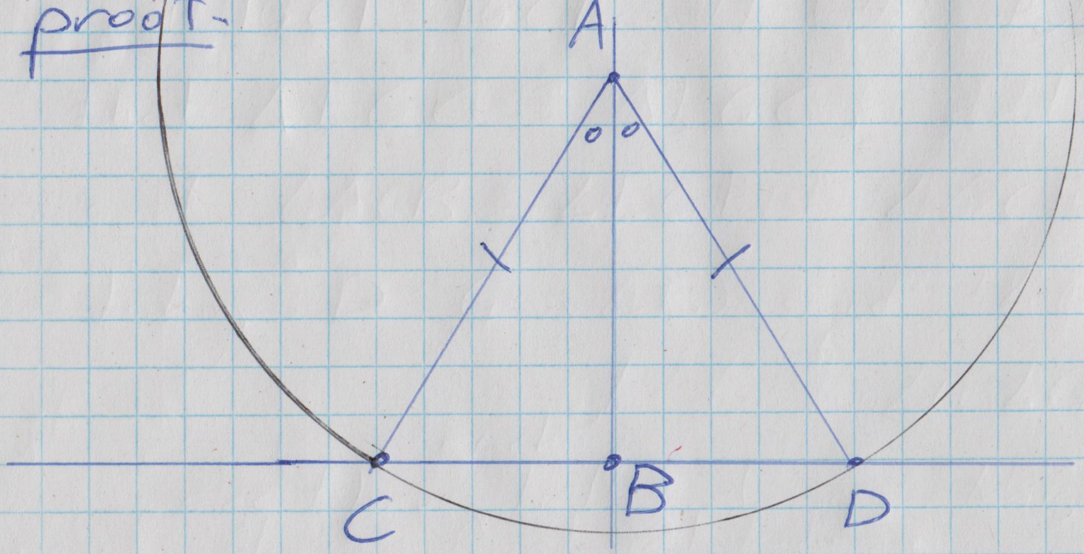
Construct an equilateral triangle on EF, with the third vertex being G. Connect C to G. Then $|EC| = |FC|$, $|GE| = |GF|$ (equilateral triangle!), and $|GC| = |GC|$. It follows by the SSS congruence criterion that $\Delta EGC \cong \Delta FGC$, and so $\angle GCE = \angle GCF$ and $\angle ACB = \angle ECF = \angle GCE + \angle GCF$, so taking $D = G$ works since $\angle GCE = \angle ACD$ and $\angle GCF = \angle BCD$ are then each half of a straight angle, so are right angles. //

Prop I-12: "To draw a straight line perpendicular to a given infinite straight line from a given point which is not on it."

[Notation: We will use lower-case letters (from the middle of the alphabet usually) like $\ell, m, n, \ldots$ to refer to (infinite straight) lines.]

ie Given a line m and a point A not on m, find a point B on m s.t. AB is perpendicular to m (ie $AB \perp m$).

proof.



Pick a point, any point!, on m, call it C.

Draw the circle with radius AC and centre A.

There are two cases:

(i) the circle intersects m only at C, [left to the reader]

(ii) the circle intersects m at some other point D as well.

case (ii) Connect A to D.

Divide $\angle CAD$ in half, with the dividing line meeting m at B. Claim: $\angle ABC = \angle ABD = \llcorner$

i.e. $AB \perp m$

Since $|AC| = |AD|$ (radii of the same circle)

and $|AB| = |AB|$ (common notion)

and $\angle CAB = \angle DAB$ (by construction),

SAS congruence criterion gives $\Delta ABC \cong \Delta ABD$.

$\Rightarrow$ But then $\angle ABC = \angle ABD$, as required. //

We'll do Prop I-13 on next time.