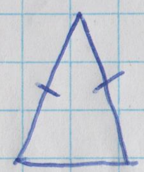
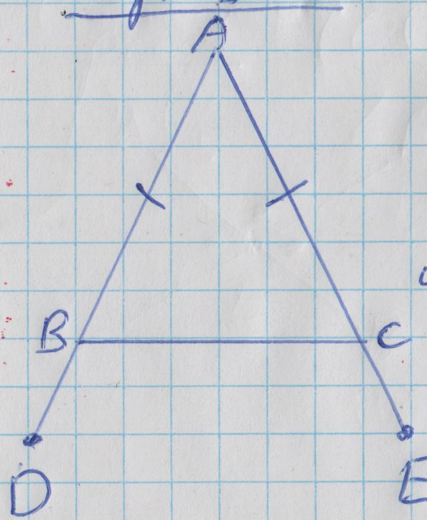
MATH 2260H - More about triangles

(Propositions I-5 through I-8.)



Recall: An isosceles triangle is one with two sides of the same length. It's traditional to use the third side as the base of the triangle.



Prop. I-5: In an isosceles triangle the angles at the base are equal to one another, and if the equal sides are extended then the angles under the base will also be equal.

i.e. If in $\triangle ABC$, we have $|AB| = |AC|$, then $\angle ABC = \angle ACB$, and if we extend AB past B to D and AC past C to E, then $\angle DBC = \angle ECB$.

Note: Look at Euclid's proof - we'll do a simplified one.

proof: ~~Apply ΔABC to ΔACB~~

Since $|AB| = |AC|$ (and $|AC| = |AB|$ ☺), and

$\angle BAC = \angle CAB$ (it's the same angle!),

it follows by the S-A-S congruence criterion (Prop. I-4) that $\Delta ABC \cong \Delta ACB$, so corresponding angle are equal, ie $\angle ABC = \angle ACB$.

Now observe that

$$\begin{aligned} \angle DBC &= \angle ABD - \angle ABC \\ &= \frown - \angle ABC \\ &= \frown - \angle ACB \\ &= \angle ACE - \angle ACB \\ &= \angle ECB. \end{aligned}$$

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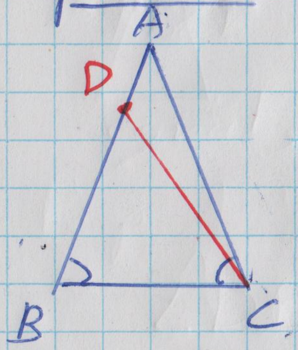
Prop. I-6. If a triangle has two equal angles, then the sides opposite those angles (ie "subtends those angles") are also equal.

proof: Suppose we are given $\triangle ABC$ with $\angle ABC = \angle ACB$.



(To show: $|AB| = |AC|$)

Assume, by way of contradiction, that $|AB| \neq |AC|$.

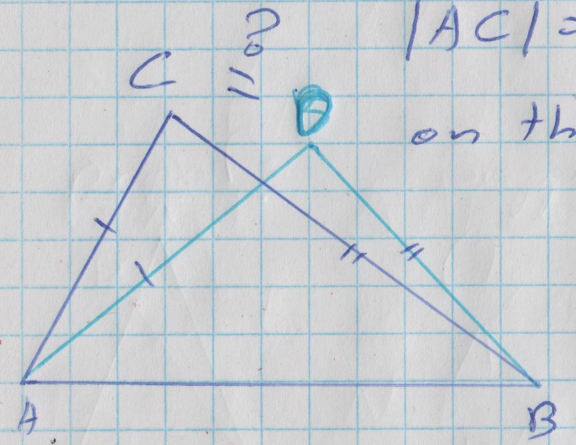
We may suppose that $|AB| > |AC|$. Then we can find a point D on AB such that $|DB| = |AC|$. [I-3]

Connect D to C and consider $\triangle DCB$:

Since $|DB| = |AC|$ and $\angle DBC = \angle ABC$ and $|BC| = |BC|$, it follows by the S-A-S congruence criterion (I-4) that $\triangle ABC \cong \triangle DCB$. Thus a smaller triangle (a sub-triangle) of $\triangle ABC$ is congruent to $\triangle ABC$, which is a contradiction.

Thus $|AB| = |AC|$, so $\triangle ABC$ is isosceles. //

Prop. I-7: You cannot find two points C and D not on a given straight line-segment AB, such that $|AC| = |AD|$ and $|BC| = |BD|$ with C and D on the same side of AB and $C \neq D$.



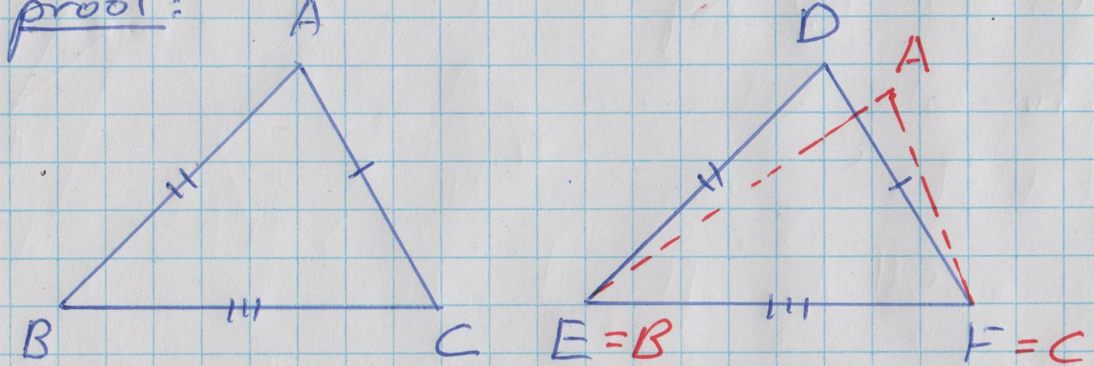
proof: See the textbook for the case drawn. What are the other cases? Prove them... (eg D is in the interior of ΔABC.) //

Q: Can you find a single proof for all the cases?

Prop I-8: (Side-Side-Side congruence criterion for triangles)

Suppose we are given two triangles, ΔABC and ΔDEF. If $|AB| = |DE|$, $|AC| = |DF|$, and $|BC| = |EF|$, then $\Delta ABC \cong \Delta DEF$.

proof:



Assume $|AB| = |DE|$,

$|AC| = |DF|$,

and $|BC| = |EF|$.

Apply ΔABC to ΔDEF, so that B falls on E, BC runs along EF, and A is on the same side of EF as D. Since BC runs along EF and $|BC| = |EF|$, C falls on F. Then the facts that $|DE| = |AB|$ and $|DF| = |AC$ mean that A is a point on the same side of EF as D, which is the same distance from each endpoint of EF as D is. By I-7, this means that A must coincide with D. But if A is on D, B is on E, and C is on F, then $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$, ie $\triangle ABC \cong \triangle DEF$. //

We have two congruence criteria for triangles so far: SAS and SSS. We will add ASA eventually. Some possible criteria don't (always) work, eg ASS.