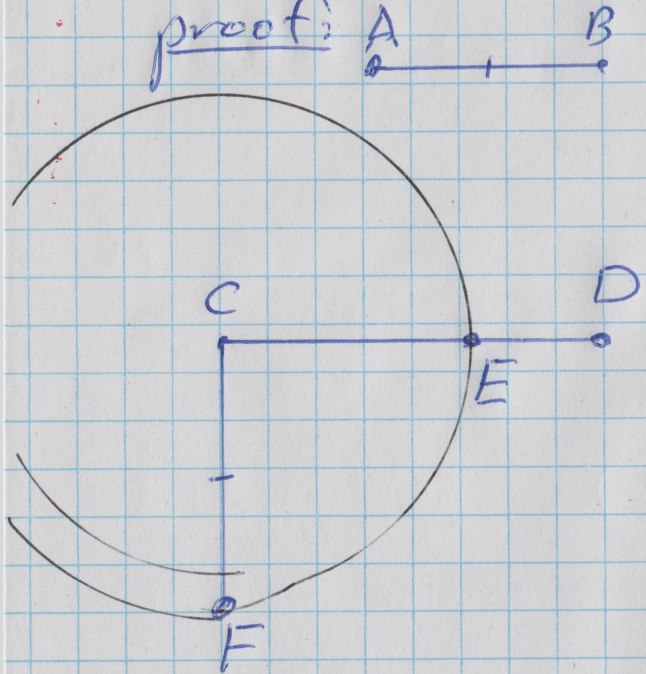
MATH 2260H - Euclidean Geometry

This time: Propositions I-3, I-4, and I-15 [done a bit early in a different way]

Prop. I-3: "For two given unequal straight lines, to cut off from the greater a straight-line equal to the lesser."

i.e. Given line segments AB and CD, with AB shorter than CD, find a point E on CD, such that $|CE| = |AB|$.

proof:



(1) By Prop. I-2, there is a point F s.t. $|CF| = |AB|$.

(2) Draw the circle with radius CF & centre C. (Post. III)

(3) Since $|CD| > |CF|$, D is outside the circle & C is inside it (centre!) By Postulate 5, it follows that the circle intersects CD at some point E.

(4) $|CE| = |CF|$ (both are radii of the circle)
$= |AB|$ from (1) Done! //

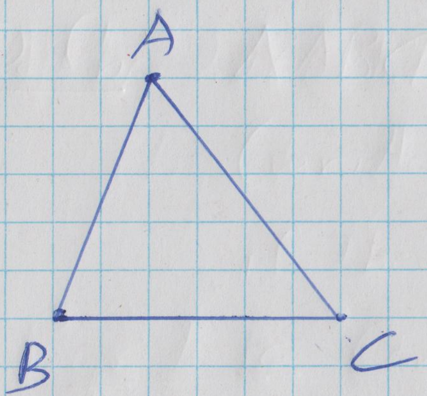
Defn: ΔABC is <u>congruent</u> to ΔDEF
(written as "$\Delta ABC \cong \Delta DEF$")
if the triangles are identical in shape and size.

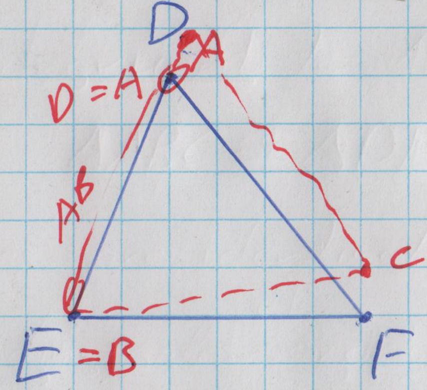
<u>ie</u> Corresponding sides are equal: $|AB| = |DE|$,
$|AC| = |DF|$,
& $|BC| = |EF|$,
and corresponding angles are equal: $\angle ABC = \angle DEF$, angle at B, angle at E
$\angle BCA = \angle EFD$,
& $\angle CAB = \angle FDE$.

<u>Prop. I-4</u> – <u>Side-Angle-Side Congruence Criterion</u> (SAS)

If two triangles have two corresponding side equal and the angle between them is equal, then the triangles are congruent.

<u>proof:</u> Suppose ΔABC and ΔDEF have
$$|AB| = |DE| \text{ and } |AC| = |DF|$$
as well as $\angle BAC = \angle EDF$.
We need to show that $|BC| = |EF|$ and
$$\angle ABC = \angle DEF \ \& \ \angle BCA = \angle EFD.$$





(1) Apply ΔABC to ΔDEF so that B falls on E, BA falls along ED, and C ends up on the same side of DE as F does. (Postulate A)

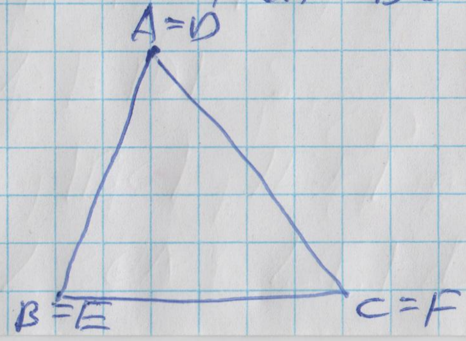
(2) Since $|AB| = |DE|$ and B falls on E and AB lies along DE, we must have $A = D$.

(3) Since A is on D and AB is on DE, and $\angle BAC = \angle DEF$, it follows that AC lies along DF.

(4) Since A is on D & AC lies along DF and $|AC| = |DF|$, it follows that C falls on F, ie $C = F$.

(5) It is *implicit* in Postulate I that the line segment joining is unique. It follows that since B is on E and C is on F, that BC coincides with EF.

Thus $|BC| = |EF|$.



(6) Since after application the vertices A, B, C coincide with the vertices D, E, F, and the sides AB, AC, BC coincide with the sides DE, DF, EF, it follows by the definition of angle ($\angle_o$) that

$\angle ABC$ coincides with $\angle DEF$ (so $\angle ABC = \angle DEF$)

and $\angle BCA$ coincides with $\angle EFD$ (so $\angle BCA = \angle EFD$).

Since corresponding sides and angles of ΔABC are equal to those of ΔDEF, it follows by definition that $\Delta ABC \cong \Delta DEF$.

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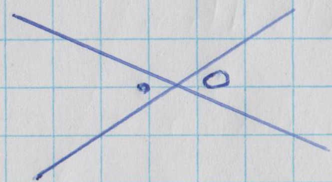
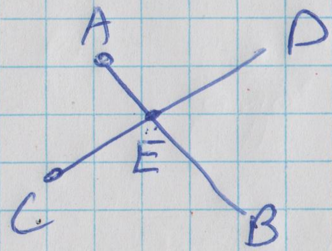
Prop. I-15 - Opposite Angles Theorem



"If two straight lines cut one another, then the vertically opposite angles are equal."

$\Rightarrow \angle\bullet = \angle\circ$

ie If line segment AB crosses line segment CD at E, then $\angle AEC = \angle BED$.



We'll use a concept that Euclid doesn't.

Def'n: $\angle ABC$ is a straight angle if A, B, C are all in a straight line (ie if you extend AB past B you will eventually encounter C)

We'll use the notation $\underline{\frown}$ for a straight angle.

Note: ~~Not~~ By the definition of right angles (~~…~~ a line falling across another making equal angles on each side of the line), $\underline{\frown} = \llcorner + \llcorner$

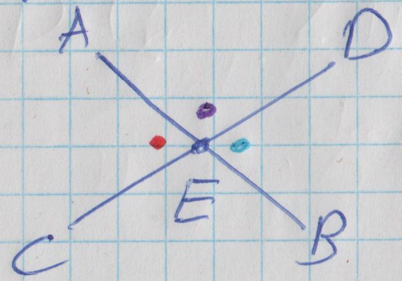
Lemma: Any two straight angles are equal.

proof: Suppose $\angle ABC$ & $\angle DEF$ are both straight angles. Then

$$\angle ABC = \llcorner + \llcorner = \angle DEF$$

since any two right angles are equal to one another by Post. IV. //

proof: (Of Prop I-15 – Opposite Angles Theorem.)



Suppose that AB crosses CD at E.
[We need to show that $\angle AEC = \angle BED$.]
Observe that $\angle CED$ and $\angle BEA$ are both straight angles. Then

$$\angle AEC = \angle CED - \angle AED = \angle BEA - \angle AED \quad \left(\text{since } \angle CED = \angle BEA \text{ by the Lemma}\right)$$

$$= \angle BED.$$

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